Theory Revision Series

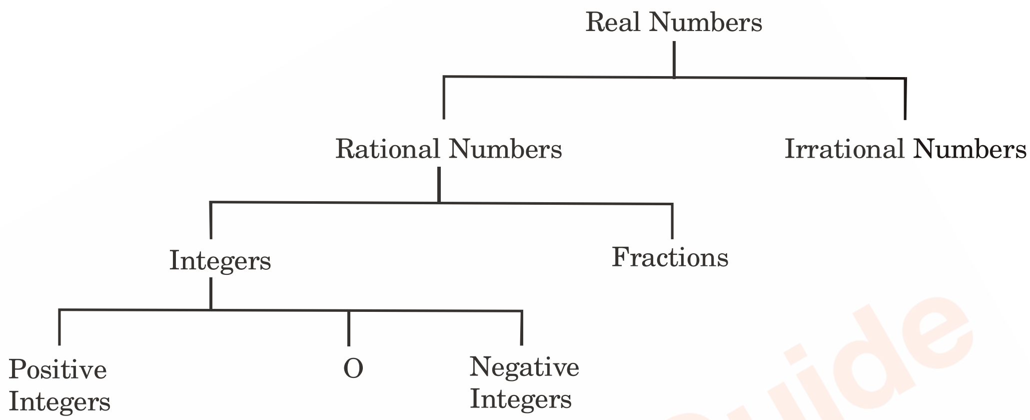
FUNCTIONS

Visit now www.iitianguide.com

FUNCTIONS

Elementary Number System

The whole of calculus is based on the concepts of real numbers. So let us briefly discuss real numbers.



Terms & their definitions

- Integers : The numbers – 4, – 3, – 2, – 1, 0, 1, 2, 3, 4 are called integers.
  i.e. set of positive **integers + zero + negative integers.**
  They are denoted by I or Z
- Natural numbers : They are a subset of integers & denoted by N
  N = {1, 2, 3, ... }
  i.e. all positive integers.
- Whole numbers : It is also a subset of integers. It contains positive integers + zero.
  Denoted by W
  W = {0, 1, 2, 3 ...}
- Zero : Zero is an integer but neither a positive nor a negative integer. But it is non-negative as well as non-positive integer.

Intervals

1. Open Interval : For two real numbers a and b, where a < b, the set of all real numbers lying strictly between a and b (i.e. not including a and b) is called an open interval.
   denoted by ( ) [round brackets]
   i.e. $a < x < b$ $\qquad$ $x \in (a, b)$

2. Closed interval : Again for same 2 real numbers, if x can take values between a and b, including a & b, then its a closed interval.

i.e. $a \le x \le b \quad x \in [a, b]$ {square brackets are used}

3. Half Open Interval : It contains both type of intervals, open, closed interval & closed open interval. In this type only one end point is included.

$a < x \le b \qquad x \in (a, b]$

$a \le x < b \qquad x \in [a, b)$

4. Infinite intervals : Before going to intervals let us discuss first about infinity, denoted by ∞.

By infinity we mean that it is a very big real number, larger than any real number but how large, it is not fixed.

When we say $x \in R$, we indirectly mean

$$-\infty < x < \infty \text{ or } x \in (-\infty, \infty)$$

coming to infinite intervals now,

whenever $\pm\infty$ is at one or both the end points we never include them; i.e.

$$-\infty < x < \infty \text{ or } x \in (-\infty, \infty)$$

round brackets

↓

Not square brackets

$-\infty < x < a$ or $x \in (-\infty, a)$

$-\infty < x \le a$ or $x \in (-\infty, a]$

$x \le a$ or $x \in (-\infty, a]$

Some Basic Definitions

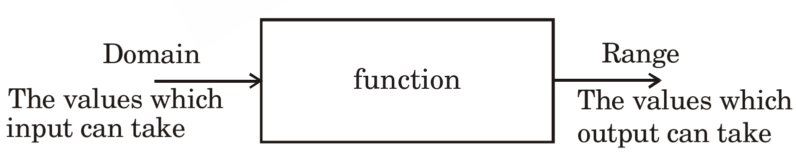
- **Domain :** For a given function $y = f(x)$, the set of values which x can take provided that for those values y is well defined, is known as Domain of the function.

  for ex. $y = \frac{1}{x}$, here x can take all real values except 0 because at $x = 0$ the value of y is invalid.

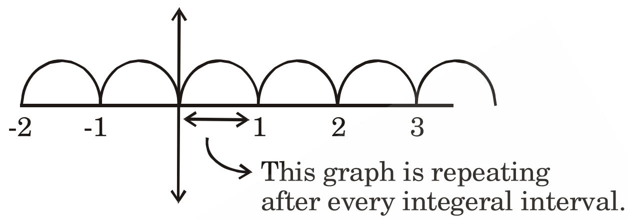
- **Range :** For a given function $y = f(x)$, the set of values which y can take, corresponding to each real number in the domain, is known as Range of function.

  for ex. $y = x^2$, here x can take all real values but y can take only positive values.

  Domain & Range can also be expressed as



- **Periodicity** : A function is said to be periodic if it repeats itself after a certain interval. for example



We just covered the basic definitions of these terms, though their properties will be discussed later on in detail.

Classification of functions :

(I) Algebraic functions :

Functions consisting of finite number of terms involving powers and roots of independent variable with the operations +, –, , ÷ are called algebraic functions.

for example : $f(x) = x + \sqrt{x}, \sqrt{x^2+1}, \frac{x+1}{x-1}$

1. Polynomial functions

$f(x) = a_0 + a_1x + a_2x^2 + \ldots a_nx_n$, where $a_0, a_1 \ldots a_n \in R$ (i.e. real constants) and n is a non negative integer, is said to be a polynomial function of degree n (given $a_n \neq 0$)

for ex. $f(x) = 3x^3 + 2x^2 + x + 1$ (polynomial of degree 3)

$f(x) = 1$ (polynomial with degree 0)

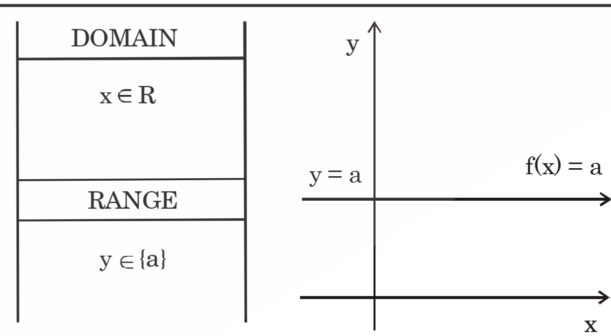
$f(x) = x^3 + \sqrt{x^2 - 1}$ (not a polynomial function)

(a) **Constant Function**

If the range of a function f consists of only one number than f is called a constant function

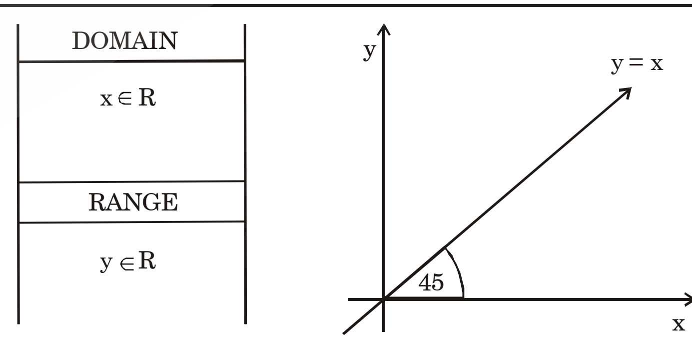
i.e. y = f (x) = a

ex. y = f (x) = 1



(b) **Identity Function**

The function y = f (x) = x is known as identify function.



2. Rational function

They are of the form $f(x)=\dfrac{P(x)}{Q(x)}$ (Q (x) ≠ 0)

where P (x) Q (x) are 2 polynomials in x & Q (x) ≠ 0 as it will make denominator 0.

Domain : Here domain is all real no. excepts when denominator is zero [i.e. Q (x) ≠ 0]

for eg. $f(x)=\dfrac{x^2-2x+1}{(x-1)(x-2)}$

here domain ∈ R – {1, 2} because at 1 & 2 denominator becomes 0.

Irrational functions

Algebraic functions consisting of non integral rational powers of x are known as irrational functions.

eg. $f(x)=x^{1/2},\ \dfrac{x^{1/2}+x^{1/3}}{\sqrt{x^2-1}},\ \dfrac{x^2+1}{x^{1/3}-1}$

Tips for Algebraic functions

1. Denominator should not be zero.
2. Expression under even root should not be negative.
3. Odd roots of any real no is defined & atleast one odd root of a real number is real.

Illustration 1

Find the domain of the function $y=\dfrac{x}{\sqrt{x^2-3x+2}}$

Solution :

Given $y=\dfrac{x}{\sqrt{x^2-3x+2}}$

for y to be valid, the value under root has to be greater than zero (here it cannot be zero because it is in denominator)

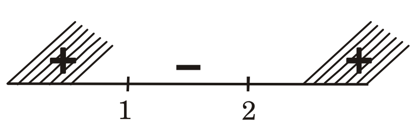
⇒ $x^2-3x+2>0$

$x^2-2x-x+2>0$

$(x-2)(x-1)>0$

⇒ $x<1$ & $x>2$

∴ domain ∈ $(-\infty, 1)\cup(2, \infty)$



TYPE (II) EXPONENTIAL & LOGARITHAMIC FUNCTION

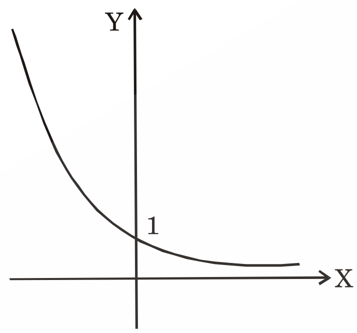
1. Exponential Function

The function $g = f(x) = a^x$, $a > 0$, $a \neq 1$ is said to be an exponential function.

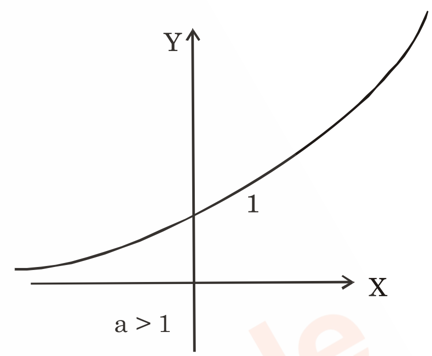
It is divided into 2 parts depending on the value of a.

for $0 < a < 1$, y decreases as x increases

$a > 1$, y increases as x increases



0 < a < 1



Shape of curve

as we can see from the graph that the value of y approaches zero but is never 0 (i.e. asymptote) and can take all positive values.

Domain	:	$x \in R$
Range	:	$y \in (0, \infty)$

2. Logaritmic Function

The function $y = f(x) = \log_a x$ is known as logarithmic function.

provided that

$x > 0$

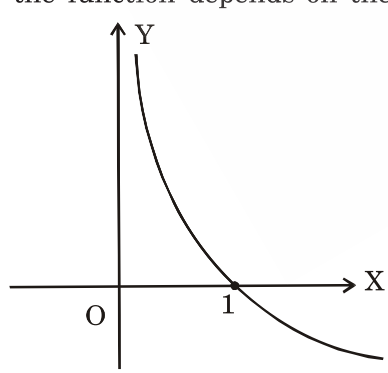
$a > 0$

and $a \neq 1$

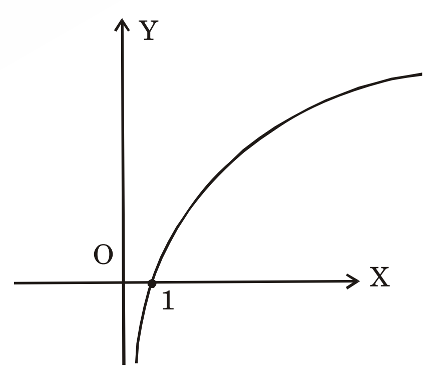
So the domain is very clear from the constraints only

Domain	:	$x \in (0, \infty)$
Range	:	$y \in (-\infty, \infty)$

Here also the function depends on the value of a.



0 < a < 1



a > 0

Properties of logarithmic functions :

1. $\log_e (ab) = \log_e a + \log_e b$

2. $\log_e \left(\frac{b}{a}\right) = \log_e b - \log_e a$

3. $\log_e a^m = m \log_e a$

4. $\log_a a = 1$

5. $\log_{b^m} a = \frac{1}{m} \log_b a$

6. $\log_b a = \frac{1}{\log_a b}$

7. $\log_b a = \frac{\log_m a}{\log_m b}$

8. $a^{\log_a b} = b$

9. $a^{\log_b c} = c^{\log_b a}$

10. If $\log_a x > \log_a y \Rightarrow \begin{cases} x > y, & \text{if } a > 1 \\ x < y, & \text{if } 0 < a < 1 \end{cases}$

11. $\log_a x = y \Rightarrow x = a^y$

12. $\log_a x > y \Rightarrow \begin{cases} x > a^y, & \text{if } a > 1 \\ x < a^y, & \text{if } 0 < a < 1 \end{cases}$

13. $\log_a x < y \Rightarrow \begin{cases} x < a^y, & \text{if } a > 1 \\ x > a^y, & \text{if } 0 < a < 1 \end{cases}$

Also, when we say $\log x = y$, then we take log with base 10.

Similarly for $\log_e x = y$ we write it as $\ln x = y$ (log with base e is also called natural log)

Illustration 2

Find domain of $f(x) = \ln(-2 + 3x - x^2)$

Solution : for f (x) to be valid the log function should be valid and for that $-x^2 + 3x - 2 > 0$

Now, $-x^2 + 3x - 2 > 0$

$\Rightarrow \quad x^2 - 3x + 2 < 0$

$\Rightarrow \quad x^2 - 2x - x + 2 < 0$

$\Rightarrow \quad (x - 2)(x - 1) < 0$

$\Rightarrow \quad x \in (1, 2)$

$\therefore \quad$ domain $\in (1, 2)$

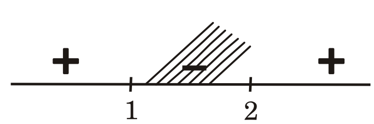


Illustration 3

Find the domain of $e^{\frac{1}{x^2-1}}$

Solution :

The function is valid for all real values except for those on which $x^2 - 1$ becomes zero.

$\therefore \quad x^2 - 1 \neq 0$

for $x = -1, 1$, $x^2 - 1$ is zero

$\therefore \quad$ domain $\in R - \{-1, 1\}$

Illustration 4

Find the domain of the following functions :

(a) $y = \sqrt{\log_{10}\left(\frac{5x - x^2}{4}\right)}$

(b) $y = \log_{10}\{\log_{10}\log_{10}\log_{10} x\}$

Solution :

(a) Given, $y = \sqrt{\log_{10}\left(\frac{5x - x^2}{4}\right)}$

for this function to be valid, the term on R.H.S. has to valid. For that to be true there are 2 conditions i.e.

1. $\frac{5x - x^2}{4} > 0$

2. $\log_{10}\left(\frac{5x - x^2}{4}\right) > 0$

First solving for part 1

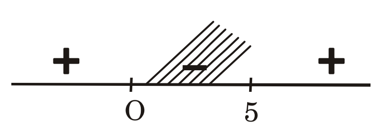
$$\frac{5x - x^2}{4} > 0$$

$\Rightarrow \quad 5x - x^2 > 0$

$\Rightarrow \quad x^2 - 5x < 0$

$\Rightarrow \quad x(x - 5) < 0$

$\Rightarrow \quad x \in (0, 5)$ .... (i)

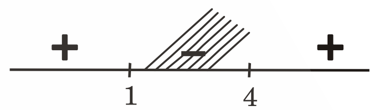


Now solving the second part

$$\log_{10}\left(\frac{5x - x^2}{4}\right) \geq 0$$

$\Rightarrow \quad \dfrac{5x - x^2}{4} \geq 10^0$

$\Rightarrow \quad \dfrac{5x - x^2}{4} \geq 1$



$\Rightarrow \quad 5x - x^2 \geq 4$

$\Rightarrow \quad x^2 - 5x + 4 \leq 0$

$\Rightarrow \quad (x - 4)(x - 1) \leq 0$

$\Rightarrow \quad x \in [1, 4]$ .... (ii)

since both conditions have to be satisfied, we have to take the intersection of (i) & (ii)

$\therefore$ from (i) & (ii)

$x \in [1, 4]$

$\therefore$ Domain $\in [1, 4]$

(b) $f(x) = \log_{10}(\log_{10}\log_{10}\log_{10} x)$

for function to be valid

$\log_{10}(\log_{10}\log_{10} x) > 0$ & $x > 0$

$\Rightarrow \quad \log_{10}(\log_{10} x) > 10^0$ & $x > 0$

$\Rightarrow \quad \log_{10}(\log_{10} x) > 1$ & $x > 0$

$\Rightarrow \quad \log_{10} x > 10^1$ & $x > 0$

$\Rightarrow \quad x > 10^{10}$ & $x > 0$

combining both we get

Domain $\in (10^{10}, \infty)$

Illustration 5

Find the domain of the function :

$$f(x) = \frac{\mathbf{1}}{\mathbf{log_{10}}(\mathbf{1} - x)} + \sqrt{x + \mathbf{1}}$$

Solution :

This question is a mix of algebraic & logarithmic functions.

Now, $f(x) = \dfrac{1}{\log_{10}(1 - x)} + \sqrt{x + 1}$

We will solve both the parts separately & then combine their results to get the final results.

$\log_{10}(1-x)$ is valid when $x < 1$

& $\dfrac{1}{\log_{10}(1-x)}$ is valid when $1 - x \neq 1$ ($\because \log_a 1 = 0$) & $x < 1$

$\Rightarrow$ $x < 1$ except $x = 0$ (from $1 - x \neq 1$ because at $x = 0$, denominator becomes 0)

$\Rightarrow$ $x \in (-\infty, 1) - \{0\}$ .... (i)

now solving the algebraic part

for $\sqrt{x+1}$ to be valid $x + 1 \geq 0$

$\Rightarrow$ $x \geq -1$ .... (ii)

combining (i) & (ii)

we get $x \in [-1, 0) \cup (0, 1)$

$\therefore$ The domain of the given functions is $[-1, 0) \cup (0, 1)$

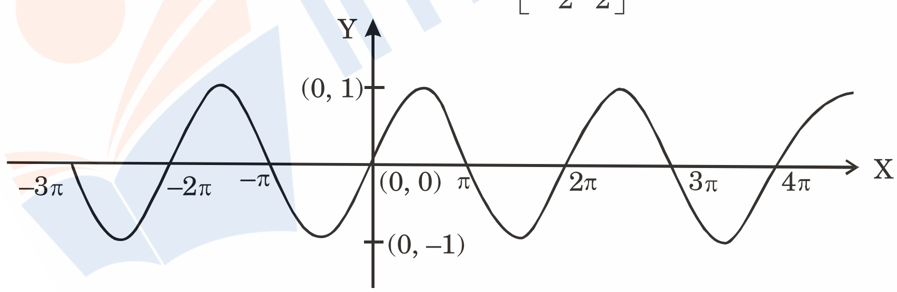
Trigonometry/circular functions :

Functions involving trigonometric ratios are called trigonometric functions.

(a) $y = f(x) = \textbf{sin}\ x$

Domain	:	$(-\infty, \infty)$;	Range	:	$[-1, 1]$;
Period	:	2π;	Nature	:	odd ;

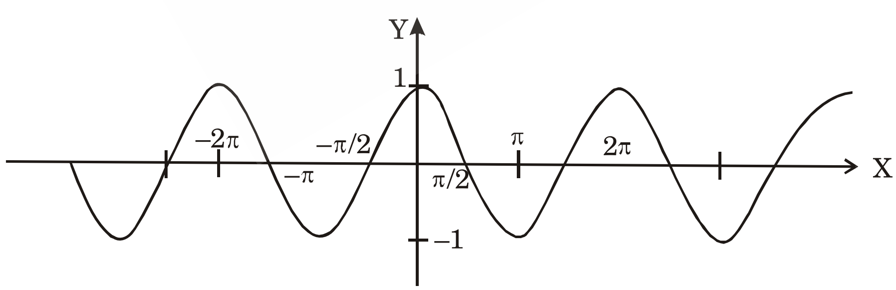
Interval in which the inverse can be obtained : $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$



(b) $y = f(x) = \textbf{cos}\ x$

Domain	:	$(-\infty, \infty)$;	Range	:	$[-1, 1]$;
Period	:	2π;	Nature	:	even;

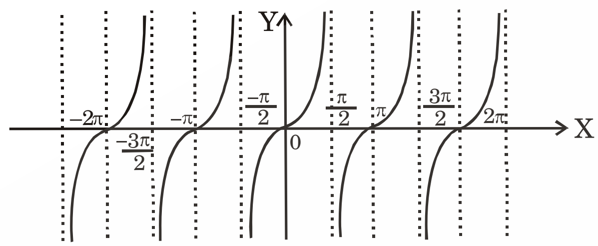
Interval in which the inverse can be obtained : $[0, \pi]$



(c) $y = f(x) = \tan x$

Domain	: $R - (2n + 1)\,\pi/2,\ n \in I$;	Range	: $(-\infty, \infty)$,
Period	: π;	Nature	: odd;

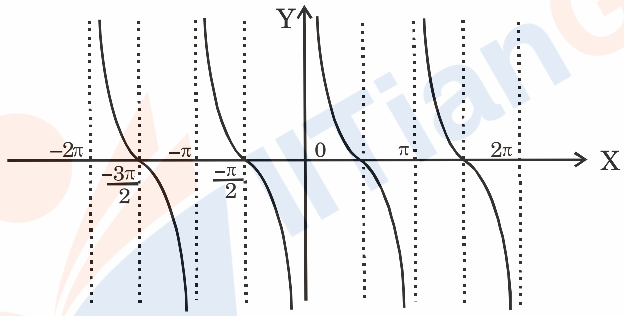
Interval in which the inverse can be obtained : $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$



(d) $y = f(x) = \cot x$

Domain	: $R - n\pi,\ n \in I$	Range	: $(-\infty, \infty)$,
Period	: π;	Nature	: odd;

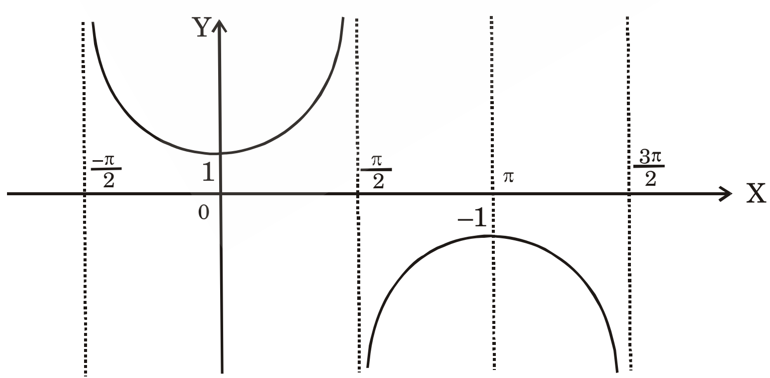
Interval in which the inverse can be obtained : $(0, \pi)$



(e) $y = f(x) = \sec x$

Domain	: $R - (2n + 1)\,\pi/2,\ n \in I$;	Range	: $(-\infty, -1] \cup [1, \infty)$
Period	: 2π;	Nature	: even ;

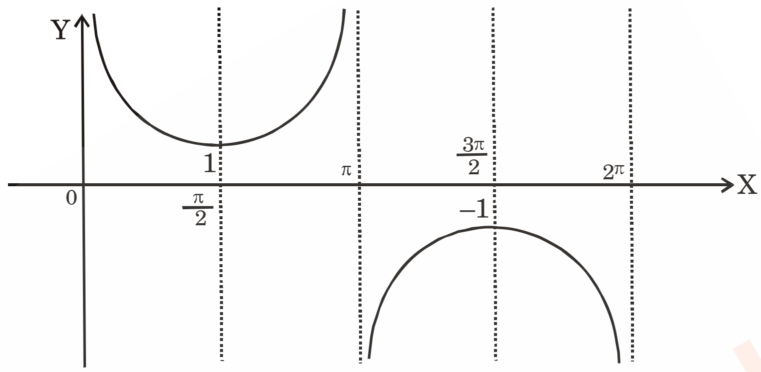
Interval in which the inverse can be obtained : $\left[0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \pi\right]$



(f) $y = f(x) = \textbf{cosec } x$

Domain	:	$R - n\pi,\ n \in I$;	Range	:	$(-\infty, -1] \cup [1, \infty)$;
Period	:	2π;	Nature	:	odd ;

Interval in which the inverse can be obtained : $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$



Inverse trigonometric/inverse circular functions :

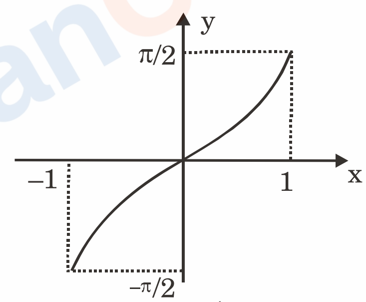
Functions involving inverse of trigonometric ratios are called inverse trigonometric or inverse circular functions.

(a) $y = f(x) = \sin^{-1} x$

Domain : $[-1, 1]$

Range : $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Nature : odd ;



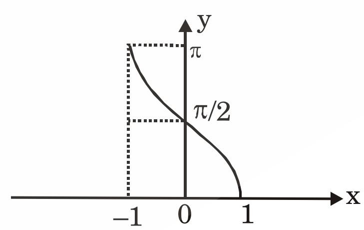
$y = \sin^{-1} x$

(b) $y = f(x) = \cos^{-1} x$

Domain : $[-1, 1]$

Range : $[0, \pi]$

Nature : neither even nor odd



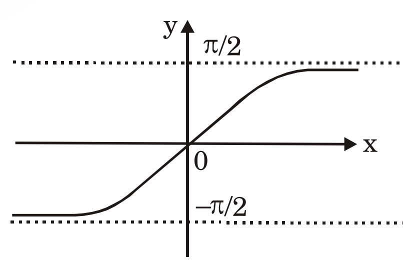
$y = \cos^{-1} x$

(c) $y = f(x) = \tan^{-1} x$

Domain : $(-\infty, \infty)$,

Range : $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Nature : odd;

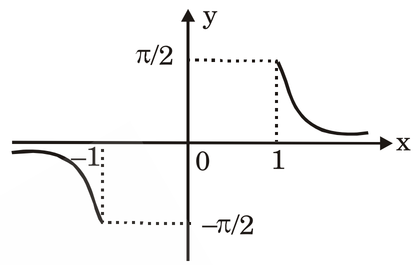


(d) $y = f(x) = \mathbf{cosec^{-1}}\, x$

Domain : $(-\infty, -1] \cup [1, \infty)$

Range : $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

Nature : odd;



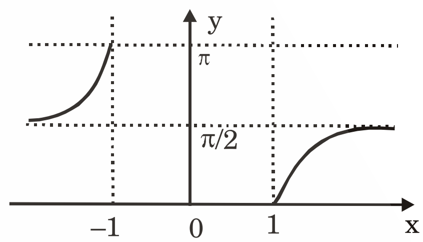
$y = \text{cosec}^{-1}\, x$

(e) $y = f(x) = \mathbf{sec^{-1}}\, x$

Domain : $(-\infty, -1] \cup [1, \infty)$

Range : $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

Nature : neither even nor odd



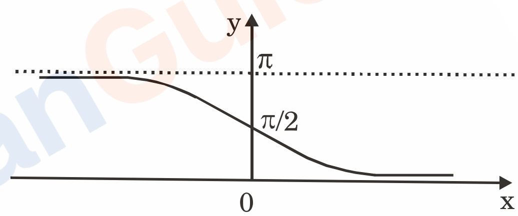
$y = \sec^{-1} x$

(f) $y = f(x) = \mathbf{cot^{-1}}\, x$

Domain : $(-\infty, \infty)$,

Range : $(0, \pi)$

Nature : neither even or odd.



$y = \cot^{-1} x$

Illustration 6

Find the domain for the following :

(a) $f(x) = \sqrt{\mathbf{cos}\,(\mathbf{sin}\, x)}$ **(b)** $y = f(x) = \mathbf{sin}^{-1}\left(\mathbf{log_2}\left(\frac{x^2}{2}\right)\right)$

Solution :

(a) $f(x) = \sqrt{\cos(\sin x)}$ is defined if

value under root is non-negative

i.e. $\cos(\sin x) \geq 0$

but we know that sin x lies between – 1 & 1

$\Rightarrow$ $-1 \leq \sin x \leq 1$

& for $[-1, 1]$ cosine function is always + ve because $\cos x \geq 0$ for $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ & $\frac{\pi}{2} > 1$

$\therefore$ $\cos(\sin x) \geq 0$ for all x

$\Rightarrow$ $x \in R$

$\therefore$ Domain $\in$ R

(b) for function, $f(x) = \sin^{-1}\left(\log_2\left(\frac{x^2}{2}\right)\right)$ to be valid

firstly $\frac{x^2}{2} > 0$ &

then $-1 < \log_2 \frac{x^2}{2} < 1$

for the first part $\frac{x^2}{2} > 0 \Rightarrow x^2 > 0$

because log can not be 0

$\Rightarrow x \in R - \{0\}$ .... (i)

for second part $-1 < \log_2 \frac{x^2}{2} < 1$

$\Rightarrow$ $2^{-1} < \frac{x^2}{2} < 2^1$

$\Rightarrow$ $\frac{1}{2} < \frac{x^2}{2} < 2$

$\Rightarrow$ $1 \leq x^2 \leq 4$

$\Rightarrow$ $-2 \leq x \leq -1$ or $1 \leq x \leq 2$ .... (ii)

though we can also solve the inequality taking cases & combing to get (ii)

from (i) & (ii)

$x \in [-2, -1] \cup [1, 2]$

Illustration 7

Find the domain of the definition of function :

(a) $f(x) = \log_{10} \sin(x - 3) + \sqrt{16 - x^2}$

(b) $y = \cos^{-1}\left(\frac{2}{2 + \sin x}\right)$

Solution :

(a) Given $f(x) = \log_{10} \sin(x-3) + \sqrt{16-x^2}$

for f(x) to be well defined

$\sin(x-3) > 0$

& we know sin x is positive in $(0, \pi)$. So generalising it, sin x will be positive in $(2n\pi + 0, 2n\pi + \pi)$ as 2π is the period of sin x.

$\Rightarrow$ $2n\pi + 0 < (x-3) < 2n\pi + \pi$

$\Rightarrow$ $2n\pi + 3 < x < (2n+1)\pi + 3$ .... (i)

for the under root part

$16 - x^2 \geq 0$

$\Rightarrow$ $x^2 - 16 \leq 0$

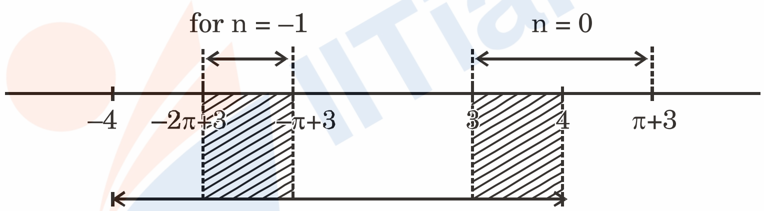
$\Rightarrow$ $(x-4)(x+4) \leq 0$

$\Rightarrow$ $x \in [-4, 4]$ .... (ii)

combining (i) & (ii)

from (i) $x \in (2n\pi + 3, (2n+1)\pi$ where $n = 0, \pm 1, \pm 2$

(ii) $x \in [-4, 4]$



$\therefore$ The common region is $(-2\pi + 3, -\pi + 3) \cup (3, 4]$

$\therefore$ Domain is $(-2\pi + 3, -\pi + 3) \cup (3, 4]$

(b) given $y = \cos^{-1}\left(\dfrac{2}{2+\sin x}\right)$

for y to be defined.

$$-1 \leq \frac{2}{2+\sin x} \leq 1$$

solving first $-1 \leq \dfrac{2}{2+\sin x}$

denominator here can never be negative or zero because sin x $\in$ [– 1, 1], cross multiplying

$\therefore \quad -(2 + \sin x) \le 2$

$\Rightarrow \quad -2 - \sin x \le 2$

$\Rightarrow \quad \sin x \ge -3$

which is true for all values of x as min. value of sin x is – 1. .... (i)

now solving the second part

$$\frac{2}{2 + \sin x} \le 1$$

$\Rightarrow \quad 2 \le 2 + \sin x$

$\Rightarrow \quad \sin x \ge 0$

$\Rightarrow \quad 2n\pi + 0 \le x \le 2n\pi + \pi$

$\Rightarrow \quad 2n\pi \le x \le (2n + 1)\pi$ .... (ii)

$\therefore$ combining (i) & (ii)

Domain $\in (2n\pi, (2n + 1)\pi)$

Some other functions

1. Absolute Value / Modulus function

By absolute / modulus function we mean only the numerical value of the function, irrespective of its sign, from origin.

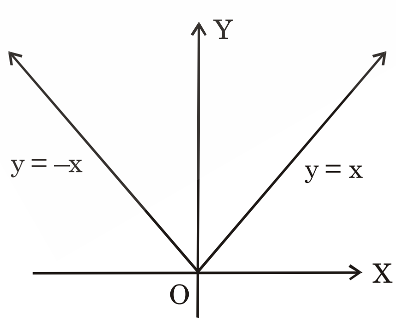
This concept is analogous to distance. We can also say that modulus function is distance with respect to origin.

Though, modulus function is defined as

$f : R \to R, f(x) = |x|$

here,

Domain :	$x \in R$	Range :	$[0, \infty)$
Period :	Non periodic	Nature :	even

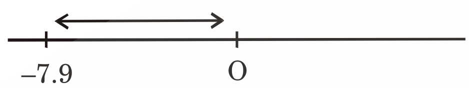


The $|x|$ can be defined as follows : $|x|$

$$|x| = \begin{Bmatrix} -x & , & x < 0 \\ x & , & x \geq 0 \end{Bmatrix}$$

for ex. $|-1| = 1$, $|2.8| = 2.8$, $|-7.9| = 7.9$

here we can understand this as distance, for ex. take $|-7.9|$



On number line the distance of – 7.9 from origin is 7.9. So the value of modulus function is 7.9

So if you are given

		if $a > 0$	$a < 0$
1.	$\lvert f(x) \rvert = a$	$\Rightarrow$ $f(x) = a$	No solution
2.	$\lvert f(x) \rvert < a$	$\Rightarrow$ $-a < f(x) < a$	No solution
3.	$\lvert f(x) \rvert > a$	$\Rightarrow$ $f(x) < -a$ or $f(x) > a$	True for all x

Basic properties of $|x|$

- $||x|| = |x|$
- $|xy| = |x||y|$
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, $y \neq 0$
- $|x + y| \leq |x| + |y|$,
- $|x - y| \geq |x| - |y|$

last two properties are intersecting ones, you can prove them by putting values.

Illustration 8

Find the domain of the following function :

(a) $\left|\frac{2}{x-4}\right| > 1$

(b) $\frac{|x|-1}{|x|-2} \geq 0$

(c) $|x - 1| + |x - 2| \geq 4$

(d) $\frac{|x+3|+x}{x+2} > 1$

Solution :

(a) We have $\left|\frac{2}{x-4}\right| > 1$

we can see that $x \neq 4$

$\Rightarrow \quad \frac{2}{|x-4|} > 1 \qquad \left\{\because \left|\frac{a}{b}\right| = \frac{|a|}{|b|} \ \& \ |2| = 2\right\}$

$\Rightarrow \quad 2 > |x - 4|$ {we can do this because mod function is always positive}

$\Rightarrow \quad |x - 4| < 2$

$\Rightarrow \quad -2 < x - 4 < 2$

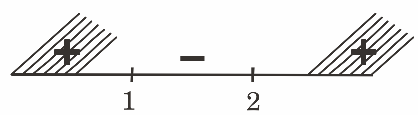
$\Rightarrow \quad 2 < x < 6$

$x \in (2, 6) - \{4\}$ (**Note :** Remember to remove 4 from domain students generally miss this step)

(b) we have, $\frac{|x| - 1}{|x| - 2} \geq 0$

let $|x| = y$

$\Rightarrow \quad \frac{y-1}{y-2} \geq 0$



$\Rightarrow \quad y > 2$ or $y \leq 1$

Note : we cannot include $y = 2$.

$\Rightarrow \quad |x| > 2$ or $|x| \leq 1$

$(x > 2$ or $x < -2)$ or $(-1 \leq x \leq 1)$

$\Rightarrow$ Domain $\in (-\infty, -2) \cup [-1, 1] \cup (2, \infty)$

(c) we have $|x - 1| + |x - 2| \geq 4$

we will solve this by finding the critical points & checking for values greater or smaller about these critical points. Here critical points are 1 & 2

$$|x - 1| = \begin{cases} (x-1) & , x \geq 1 \\ -(x-1) & , x < 1 \end{cases}$$

& $$|x - 2| = \begin{cases} (x-2) & , x \geq 2 \\ -(x-2) & , x < 2 \end{cases}$$

we can divide the values in 3 region i.e. < 1, between 1 & 2, & greater than 2.

Case 1 : when $-\infty < x < 1$

i.e. in this region $|x-1| = -(x-1)$

& $|x-2| = -(x-2)$

$|x-1| + |x-2| \geq 4$

$\Rightarrow \quad -(x-1)-(x-2) \geq 4$

$\Rightarrow \quad -2x+3 \geq 4$

$\Rightarrow \quad 2x-1 \leq 0$

$\Rightarrow \quad x \leq -½$ .... (i)

Case 2 : when $1 \leq x \leq 2$

here in this region $|x-1| = x+1$

& $|x-2| = -(x-2)$

$|x-1| + |x-2| \geq 4$

$\Rightarrow \quad x-1-(x-2) \geq 4$

$\Rightarrow \quad 1 \geq 4$

$\Rightarrow$ no solution for this solution .... (ii)

Case 3 : when $x > 2$

here both are positive i.e. $|x-1| = x-1$

& $|x-2| = x-2$

$|x-1| + |x-2| \geq 4$

$\Rightarrow \quad x-1 + x-2 \geq 4$

$\Rightarrow \quad 2x-3 \geq 4$

$\Rightarrow \quad x \geq 7/2$ .... (iii)

$\therefore$ combining (i), (ii) & (iii)

Domain $\in \left(-\infty, \frac{-1}{2}\right] \cup \left[\frac{7}{2}, \infty\right)$

(d) $\frac{|x+3|+x}{x+2} > 1$

$\Rightarrow \quad \frac{|x+3|+x}{x+2} - 1 > 0$

$\Rightarrow \quad \frac{|x+3|+x-(x+2)}{x+2} > 0$

$\Rightarrow \quad \frac{|x+3|-2}{x+2} > 0$ .... (i)

we know that $x \neq (-2)$

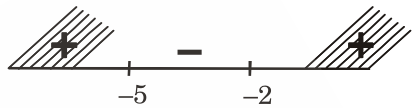
now 2 cases arise for $|x+3|$

Case 1 : $x + 3 < 0$ $\Rightarrow$

$|x + 3| = -(x - 3)$

putting these values in (i)

$\Rightarrow \quad \dfrac{-(x+3)-2}{x+2} > 0$



$\Rightarrow \quad \dfrac{-x-5}{x+2} > 0$

$\Rightarrow \quad \dfrac{x+5}{x+2} < 0$

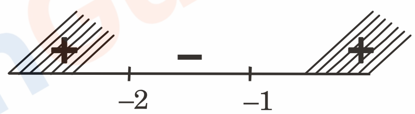
[Now here note that we are getting answer $(-5, -2)$ but do not forget that this case is for $x < -3$.

So the answer for this part is $(-5, -3]$ .... (ii)

Case : $x + 3 \geq 0$, for $x \geq -3$

$\Rightarrow \quad |x + 3| = x + 3$

i.e. $\quad \dfrac{(x+3)-2}{x+2} \geq 0$



$\dfrac{x+1}{x+2} \geq 0$

$\Rightarrow \quad x \in (-\infty, -2] \cup [-1, \infty)$

but again this region is for $x \geq -3$

$\Rightarrow \quad x \in [-3, -2) \cup [-1, \infty)$ .... (iii)

combining (ii) & (iii)

Domain $\in (-5, -3) \cup [-3, -2) \cup [-1, \infty)$

$\in [-5, -2) \cup [-1, \infty)$

Greatest Integer function/Step function

The function $y = [x]$ is known as greatest Integer function & is defined as greatest integer less than equal to x.

i.e. $\quad y = [x] = a$ if $a \leq x < a + 1$

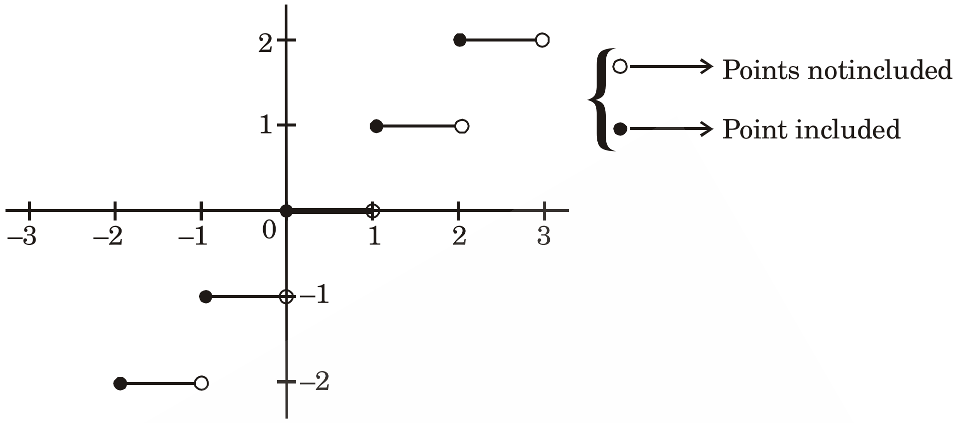
for example

$[2.3] = 2, [5.9] = 5, [7] = 7,$

take special care of negative values

$[-7.9] = -8, [-5] = -5$

$y = f(x) = [x]$



Domain	:	R	Range	:	I (i.e. set of ingtegers)
Period	:	Non-periodic	Nature	:	neither odd/nor even

The function is called step function as we can see from the graph that it follows a step like curve

Fraction part of x :

for example if x = 5.9

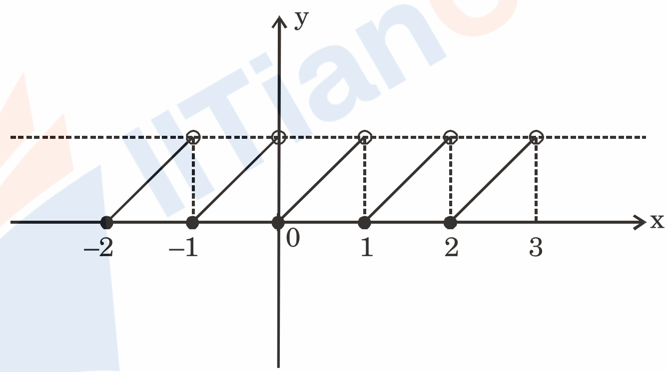
then fractional part of x is .9

It is denoted by {x}

Thus, {x} = x – [x]

& $0 \le \{x\} < 1$

$y = f(x) = \{x\}$



Domain	:	R	Range	:	[0, 1)
Period	:	1	Nature	:	neither even nor odd

Some properties of greatest integer function & fractional part

(i) $[[x]] = [x]$

(ii) $[x + n] = [x] + n$, if n is an integer

(iii) $[\{x\}] = 0$, $\{[x]\} = 0$

(iv) $[x] + [-x] = \begin{Bmatrix} 0 & ; \text{if } x \in \text{integer} \\ -1 & ; \text{if } x \notin \text{integer} \end{Bmatrix}$

(v) $\{x\} + \{-x\} = \begin{Bmatrix} 0 & ; \text{if } x \in \text{integer} \\ +1 & ; \text{if } x \notin \text{integer} \end{Bmatrix}$

(vi) $[-x] = \begin{cases} -[x] & ; \text{ if } x \in \text{integer} \\ -[x]-1 & ; \text{ if } x \notin \text{integer} \end{cases}$

(vii) $[x + y] = \begin{cases} [x] + [y] & , \bullet \text{ either one of } x \text{ or } y \text{ is integer} \\ & \bullet \{x\} + \{y\} < 1 \\ [x] + [y] + 1 & \text{ when } \{x\} + \{y\} \geq 1 \end{cases}$

(viii) $\left[\frac{[x]}{n}\right] = \left[\frac{x}{n}\right]$, $n \in N$

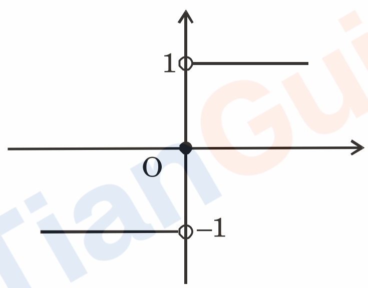
(ix) $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \ldots\ldots + \left[x + \frac{n-1}{n}\right] = [nx]$, $n \in N$

(c) Signum function

The function is defined as

$y = f(x) = \text{sgm}(x)$

$$y = \text{sgn}(x) \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$



Domain	: R	Range	: {– 1, 0, 1}
Period	: non-periodic	Nature	: odd

Illustration 9

Find the domain of the following :

(a) $[x]^2 - 3[x] + 2 \leq 0$

(b) $4[x] = x + \{x\}$

Solution :

(a) given $[x]^2 - 3[x] + 2 \leq 0$

$\Rightarrow \quad ([x] - 1)([x] - 2) \leq 0$

$\Rightarrow \quad 1 \leq [x] \leq 2$

Here the value of greatest integer function is 1 & 2.

for value 2, x can lie b/w 2 & 3

$\therefore$ Domain $\in [1, 3)$

> TIP : if $[x] \in n$ where $n \in I$
> $x \in [n, n + 1)$

(b) given the function, $4[x] = x + \{x\}$

but $x = [x] + \{x\}$

putting this

$$4[x] = [x] + \{x\} + \{x\}$$

$$4[x] = [x] + 2\{x\}$$

$$3[x] = 2\{x\}$$

$\Rightarrow$ $\{x\} = \frac{3}{2}[x]$ or $[x] = \frac{2}{3}\{x\}$

we know $0 \le \{x\} < 1$

$\Rightarrow$ $0 \le [x] < \frac{2}{3}$

$\Rightarrow$ $[x] = 0$

$\Rightarrow$ $0 \le x < 1$ .... (i)

but $\{x\} = \frac{3}{2}[x]$

& for $0 \le x < 1$, $[x] = 0$

$\Rightarrow$ $\{x\} = 0$ . ... (ii)

combining (i) & (ii)

The only value to satisfy the question is $x = 0$

Illustration 10

Find the domain of the following :

(a) $x^2 - 4x + [x] + 3 = 0$ **(b)** $f(x) = \frac{\sin[x-2]}{[x-2][x+3]}$

Solution : (a) given that $x^2 - 4x + [x] + 3 = 0$

$\Rightarrow$ $x^2 - 4x + (x - \{x\}) + 3 = 0$

$\Rightarrow$ $x^2 - 3x + 3 = \{x\}$

$\Rightarrow$ $0 \le x^2 - 3x + 3 < 1$

But $x^2 - 3x + 3 = x^2 - 3x + \frac{9}{4} + 3 - \frac{9}{4}$

$$= \left(x - \frac{3}{2}\right)^2 + \frac{3}{4}$$

which is always greater than 0

solving now $x^2 - 3x + 3 < 1$

$\Rightarrow \quad x^2 - 3x + 2 < 0$

$\Rightarrow \quad (x - 2)(x - 1) < 0$

$\Rightarrow \quad x \in (1, 2)$ .... (i)

$\Rightarrow \quad [x] = 1$

putting back the value in the original equation

$\Rightarrow \quad x^2 - 4x + 4 = 0$

$\Rightarrow \quad (x - 2)^2 = 0$

$\Rightarrow \quad x = 2$

but from (i) $x \in (1, 2)$

$\therefore$ Thus equation has no solution.

(b) Given that $f(x) = \dfrac{\sin[x-2]}{[x-2][x+3]}$

there, there are 2 critical points -3 & 2 which divides the number line in 3 region.

1. $-3 \le x < 2$

   $[x - 2]$ give negative values but there are no problem

   but for $[x + 3]$, if $x \in (-2, -3]$

   the function becomes invalid for these values as denominator becomes 0.

   $\therefore \quad x \in [-2, 2)$ .... (i)

2. $x < -3$ in this region there is no problem as the function is valid for all values

   $\Rightarrow x \in (-\infty, -3)$ .... (ii)

3. $x \ge 2$ for $x \in [2, 3)$, the function $[x - 2]$ returns 0, which makes the function invalid but for $x \ge 3$ there is no problem.

   $\therefore \quad x \in [3, \infty)$ .... (iii)

   combining (i), (ii) & (iii)

   $x \in (-\infty, -3) \cup [-2, 2) \cup [3, \infty)$

DOMAIN

Working Rule :

In order to find the domain of the function defined by y = f (x), find the real values of x for which y is defined i.e. y is real. The set of all these values of x will be the domain.

Use the following informations whichever is required.

1. (a) sin x and cos x are defined for all real x.

(b) tan x and sec x are not defined at odd multiples of $\frac{\pi}{2}$

(c) cot x and cosec x are not defined at multiples of π.

(d) $\left.\begin{matrix} -1 \le \sin x \le 1 \\ -1 \le \cos x \le 1 \end{matrix}\right\} \left.\begin{matrix} -\infty < \tan x < \infty \\ -\infty < \cot x < \infty \end{matrix}\right\} \left.\begin{matrix} \sec x \le -1 \text{ or } \sec x \ge 1 \\ \operatorname{cosec} x \le -1 \text{ or } \operatorname{cosec} x \ge 1 \end{matrix}\right\}$

2. (a) $\sin^{-1}x$ and $\cos^{-1} x$ are defined if and only if $-1 \le x \le 1$.

(b) $\tan^{-1} x$ and $\cot^{-1}$ are defined for all real x.

(c) $\sec^{-1} x$ and $\operatorname{cosec}^{-1} x$ are defined if and only if $x \le -1$ or $x \ge 1$.

(d) $\left.\begin{matrix} -\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2} \\ -\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2} \\ -\frac{\pi}{2} \le \operatorname{cosec}^{-1} x \le \frac{\pi}{2} \\ \text{But } \operatorname{cosec}^{-1} x \ne 0 \end{matrix}\right\} \begin{matrix} 0 \le \cos^{-1} x \le \pi \\ 0 < \cot^{-1} x < \pi \\ 0 \le \sec^{-1} x \le \pi \\ \text{But } \sec^{-1} x \ne \frac{\pi}{2} \end{matrix}$

3. (a) $\log_b a$ is defined if and only if $a > 0$, $b > 0$ and $b \ne 1$ & $a \ne 0$

(b) $\log_b a > c \Leftrightarrow \left\{\begin{matrix} a > b^c, \text{ if } b > 1 \\ a < b^c, \text{ if } b < 1 \end{matrix}\right.$

(c) If $a > 0$, then a^x is defined for all real x.

4. For any function Denominator should never be zero.

5. For an algebraic function.

(a) Denominator should never be zero.

(b) an expression under even root should be ≥ 0.

6. Use sign scheme for a rational function if required.

7. Find solution of Trigonometrial inequality if required

Some useful tips to find domain

(a) Domain of $(f(x) \pm g(x))$ = Domain of $f(x) \cap$ Domain of $g(x)$

(b) Domain of $(f(x) . g(x))$ = Domain of $f(x) \cap$ Domain of $g(x)$

(c) Domain of $\left(\frac{f(x)}{g(x)}\right)$ = Domain of $f(x) \cap$ Domain of $g(x)$ $\{x;\ g(x) \neq 0\}$

(d) Domain of $\sqrt{f(x)}$ = Domain of $f(x)$ such that $f(x) \geq 0$

(e) Domain of $\log_a f(x)$ = Domain of $f(x)$ such that $f(x) > 0$

Illustration 11

If [x] denotes the integral part of x, find the domain of definition of the function.

$$f(x) = \frac{\sec^{-1} x}{\sqrt{x - [x]}}$$

Solution : For f (x) to be defined,

(i) $x - [x] > 0 \Rightarrow x > [x]$

$\Rightarrow [x] < x$

$\Rightarrow x \neq$ an integer ... (1)

(ii) $\sec^{-1} x$ should be defined,

$\Rightarrow x \leq -1$ or $x \geq 1$ .... (2)

From (1) and (2), common values of x are given by

$(-\infty < x < -1$ or $1 < x < \infty)$ and $x \notin I$

$\therefore$ Domain = $R - ((-1, 1) \cup I)$

Illustration 12

Find the domain of the function

$f(x) = \log\{ax^3 + (a + b)x^2 + (b + c)x + c\}$, **if** $b^2 - 4ac < 0$ **and** $a > 0$

Solution : Given, $f(x) = \log\{ax^3 + (a + b)x^2 + (b + c)x + c\}$ .... (i)

For f (x) to be defined,

$ax^3 + (a + b)x^2 + (b + c)x + c > 0$

$\Rightarrow (ax^3 + bx^2 + cx) + (ax^2 + bx + c) > 0$

$\Rightarrow x(ax^2 + bx + c) + ax^2 + bx + c > 0$

$\Rightarrow (x + 1)(ax^2 + bx + c) > 0$

$\Rightarrow x + 1 > 0$ $[\because b^2 - 4ac < 0$ and $a > 0$

$\therefore ax^2 + bx + c > 0$ for all real x]

$\Rightarrow x > -1$

Hence domain of f = $(-1, \infty)$

Illustration 13

Find the domain of definition of the following functions :

(i) $f(x) = \sin^{-1}(x^2 - 4x + 4)$ **(ii)** $f(x) = \sqrt{\log_{\frac{1}{2}}(2x - 3)}$

(iii) $f(x) = \sqrt{\dfrac{(x-1)(x+2)}{(x-3)(x-4)}}$ **(iv)** $f(x) = \cos^{-1}[2x^2 - 3]$

([.] denotes the greatest integer function).

Solution :

(i) For f (x) to be defined $-1 \le x^2 - 4x + 4 \le 1$

$\Rightarrow \quad -1 \le (x-2)^2 \le 1 \Rightarrow |x - 2| \le 1 \Rightarrow -1 \le x - 2 \le 1 \Rightarrow 1 \le x \le 3$

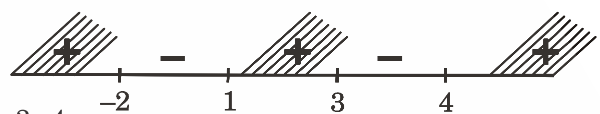
Hence the domain of definition of f(x) is the set $x \in [1, 3]$.

(ii) For f(x) to be defined $\log_{1/2}(2x - 3) \ge 0$

$\Rightarrow \quad 2x - 3 \le 1 \Rightarrow x \le 2$ .... (1) $[\log_a b \ge 0$ when

Also $2x - 3 > 0 \Rightarrow x > \dfrac{3}{2}$ .... (2) $0 < a < 1, b \le 1]$

Combining (1) and (2) we get the required values of x. Hence the domain of definition of f(x) is the set $\left(\dfrac{3}{2}, 2\right]$



(iii) For f(x) to be defined $\dfrac{(x-1)(x+2)}{(x-3)(x-4)} \ge 0$ and $x \ne 3, 4$.

By wayy-curve method the domain of definition of f(x) is the set

$x \in (-\infty, -2] \cup [1, 3) \cup (4, \infty)$

(iv) For f(x) to be defined $-1 \le [2x^2 - 3] \le 1$

$\Rightarrow \quad -1 \le 2x^2 - 3 < 2 \Rightarrow 2 \le 2x^2 < 5 \Rightarrow 1 \le x^2 < 5/2$

If $x^2 \ge 1$ then $x \in (-\infty, -1] \cup [1, \infty)$ ..... (1)

If $x^2 < \dfrac{5}{2}$ then $x \in \left(-\sqrt{\dfrac{5}{2}}, \sqrt{\dfrac{5}{2}}\right)$ ..... (2)

Combining (1) and (2), $x \in \left(-\sqrt{\dfrac{5}{2}} - 1\right] \cup \left[1, \sqrt{\dfrac{5}{2}}\right)$

which is the domain of definition of f(x).

RANGE

Working Rule :

First of all, find the domain.

1. If domain does not contain an interval, find the value of x putting the values of x from the domain. The set of all these values of y will be the range.
2. If function is continuous and domain contains only finite intervals, find the least and greatest values of y for values of x in the domain. If α and β be the least and greatest values of y for values of x in the domain, then range f = $[\alpha, \beta]$. In order to find the least and greatest values of y, write down the sign scheme for $\frac{dy}{dx}$.

   This method of finding the range of f (x) can also be used when domain is R or contains an infinite interval provided f (x) is continuous in the domain.
3. If domain is R or the set of all real numbers except a few points, then express x in terms of y and from this, find the value of y for which x is real and belongs to the domain. The set of all these values of y will be the range. But if domain does not contain some points say α and β, then find y when $x = \alpha$, and $x = \beta$ and exclude these values of y.

Illustration 14

Find domain and range of the function $y = \log_e (3x^2 - 4x + 5)$.

Solution :

y is defined if $3x^2 - 4x + 5 > 0$

where $D = 16 - 4(3)(5) = -44 < 0$

and coefficient of $x^2 = 3 > 0$

Hence, $(3x^2 - 4x + 5) > 0 \;\forall\; x \in R$

Thus, Domain is $\in R$

Now, $y = \log_e (3x^2 - 4x + 5)$

We have $3x^2 - 4x + 5 = e^y$

or $3x^2 - 4x + (5 - e^y) = 0$

Since, x is real thus, discriminant ≥ 0

$\Rightarrow \quad 12\, e^y \geq 44$

So, $y \geq \log\left(\frac{11}{3}\right)$

Hence, range is $\left[\log\left(\frac{11}{3}\right), \infty\right)$

Illustration 15

Find the range of the function $f(x) = \left[\ln\left(\sin^{-1}\sqrt{x^2+x+1}\right)\right]$, **where [.] denotes the greatest integer function.**

Solution :

Given, $f(x) = \left[\log_e\left(\sin^{-1}\sqrt{x^2+x+1}\right)\right]$ .... (1)

Domain of f :

For f (x) to be defined,

(i) $x^2 + x + 1 \geq 0 \Rightarrow -\infty < x < \infty$ ... (A)

(ii) $-1 \leq \sqrt{x^2+x+1} \leq 1 \Rightarrow x^2 + x + 1 \leq 1 \left[\because x^2+x+1 > 0\right]$

$\Rightarrow x^2 + x + 1 \leq 1 \Rightarrow x^2 + x \leq 0$

$\Rightarrow -1 \leq x \leq 0$ ... (B)

(iii) $\sin^{-1}\sqrt{x^2+x+1} > 0$

$\Rightarrow \sin^{-1}\sqrt{x^2+x+1} > \sin^{-1} 0$

$\Rightarrow \sqrt{x^2+x+1} > 0$ [$\therefore \sin^{-1} x$ is an increasing function]

$\Rightarrow x^2 + x + 1 > 0$

$\Rightarrow -\infty < x < \infty$ ... (C)

From (A), (B) and (C), $-1 \leq x \leq 0$

$\therefore$ Domain f = [– 1, 0]

Range of f :

Least value of $x^2 + x + 1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}$ at $x = \frac{-1}{2}$

$\therefore$ least value of $\sqrt{x^2+x+1} = \frac{\sqrt{3}}{2}$

Also $\sqrt{x^2+x+1} \leq 1$, $\left[\text{Since } \sin^{-1}\sqrt{x^2+x+1} \text{ should be defined}\right]$

Thus $\frac{\sqrt{3}}{2} \le \sqrt{x^2 + x + 1} \le 1$

$$\Rightarrow \quad \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \le \sin^{-1}\sqrt{x^2 + x + 1} \le \sin^{-1}(1)$$

[$\because$ $\sin^{-1} x$ is an increasing function]

$$\Rightarrow \quad \frac{\pi}{3} \le \sin^{-1}\sqrt{x^2 + x + 1} \le \frac{\pi}{2}$$

$$\Rightarrow \quad \log_e\left(\frac{\pi}{3}\right) \le \log_e\left(\sin^{-1}\sqrt{x^2 + x + 1}\right) \le \log_e\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad 0 < \log_e\left(\sin^{-1}\sqrt{x^2 + x + 1}\right) < 1 \quad \left[\because \log_e\left(\frac{\pi}{2}\right) < 1\right] \ \& \ \left[\log_e\frac{\pi}{3} > 0\right]$$

$$\Rightarrow \quad \left[\log_e\left(\sin^{-1}\sqrt{x^2 + x + 1}\right)\right] = 0$$

$\therefore$ f (x) = 0 for all $x \in [-1, 0]$

Hence range f = {0}

Note : Here f is a many-one function.

Illustration 16

Find the range of $f(x) = \log_e \frac{1}{[\cos x] - [\sin x]}$ **where [x] denotes the integral part of x.**

Solution :

Given, $f(x) = \log_e \frac{1}{[\cos x] - [\sin x]}$ ... (1)

Domain of f :

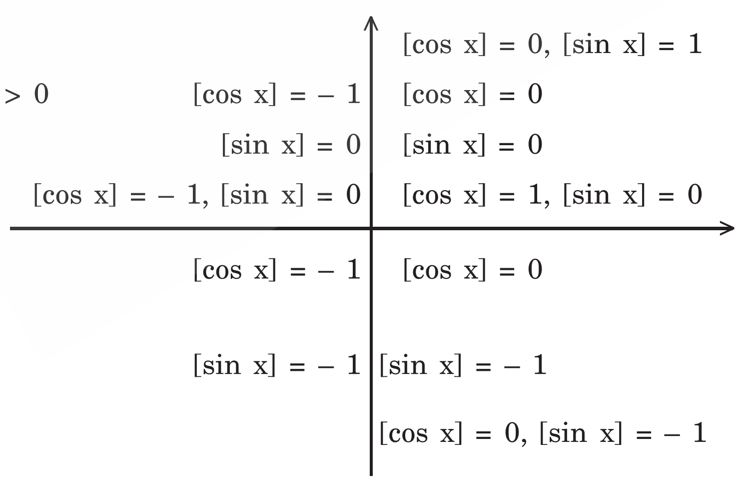
For f (x) to be defined,

$[\cos x] - [\sin x] > 0$

$\Rightarrow \quad [\cos x] > [\sin x]$

$\Rightarrow \quad -\frac{\pi}{2} \le x \le 0$

Domain = $\left[-\frac{\pi}{2}, 0\right]$



Range of f : In $\left[-\frac{\pi}{2}, 0\right]$, [cos x] – [sin x] = 1

$\therefore$ from (1), f (x) = $\log_e 1 = 0$

Hence range f = {0}

Note : Here f is a many-one function.

Illustration 17

Find the range of the following functions :

(i) $f(x) = \ln\sqrt{x^2 + 4x + 5}$

(ii) $f(x) = 3\sin x + 8\cos\left(x - \frac{\pi}{3}\right) + 5$

(iii) $f(x) = \sin^{-1}\left[\frac{1}{2} + x^2\right]$

(iv) $f(x) = \frac{2x - 2}{x^2 - 2x + 3}$

([.] denotes the greatest integer function)

Solution :

(i) Here f(x) = $\ln\sqrt{x^2 + 4x + 5} = \ln\sqrt{(x+2)^2 + 1}$

i.e. $x^2 + 4x + 5$ takes all values in $[1, \infty) \Rightarrow$ f(x) will take all values in $[0, \infty)$.

Hence range of f (x) is $[0, \infty)$.

(ii) Here f (x) = 3 sin x + 8 cos $\left(x - \frac{\pi}{3}\right) + 5$

= 3 sin x + 4 (cos x + $\sqrt{3}$ sin x) + 5 = (3 + 4 $\sqrt{3}$) sin x + cos x + 5.

Put 3 + 4 $\sqrt{3}$ = r cos θ and 4 = r sin θ so that

$r = \sqrt{73 + 24\sqrt{3}}$ and $\theta = \tan^{-1}\frac{4}{3 + 4\sqrt{3}}$ $\Rightarrow$ $f(x) = \sqrt{73 + 24\sqrt{3}}$ sin (x + θ) + 5

$\Rightarrow$ Range of f(x) is $\left[5 - \sqrt{73 + 24\sqrt{3}}, 5 + \sqrt{73 + 24\sqrt{3}}\right]$

(iii) Here $f(x) = \sin^{-1}\left[\frac{1}{2} + x^2\right]$

For any value of x, $\left[\frac{1}{2} + x^2\right]$ is a non-negative integer and $\sin^{-1} x$ is defined only for two non-negative integers 0 and 1.

$\Rightarrow$ the range of $f = \left\{0, \frac{\pi}{2}\right\}$

(iv) Here $f(x)=\dfrac{2x-2}{x^2-2x+3}$

Let y = f (x) i.e. $y=\dfrac{2x-2}{x^2-2x+3} \Rightarrow yx^2-2(y+1)x+3y+2=0$

which is a quadratic in x. For above quadratic to have real roots $\Delta \geq 0$

$\Rightarrow \quad 4(y+1)^2-4y(3y+2) \geq 0$

$\Rightarrow \quad y^2 \leq \dfrac{1}{\sqrt{2}} \quad y \leq \dfrac{1}{\sqrt{2}}$

Hence the range of f(x) is $\left[-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right]$.

Nature of Function

Working Rule :

1. (a) A function f (x) is odd if f (– x) = – f (x) i.e. f (– x) + f (x) = 0

(b) A function f (x) is even if f (– x) i.e. f (– x) – f (x) = 0

(c) Graph of an even function is symmetrical about y-axis.

(d) Graph of an odd function has the property that its part in first and third quadrants are symmetric about the origin and its part in second and fourth quadrants are symmetrical about the origin.

2. Properties of odd and even functions.

(a) a constant function is an even function

(b) a zero function is both an odd and an even function.

(c) For two functions, the following are the rules for their respective operations.

Functions	Sum	Difference	Product	Division
even – even	even	even	even	even
even – odd	neither even nor odd	neither even nor odd	odd	odd
odd – even	neither even nor odd	neither even nor odd	odd	odd
odd – odd	odd	odd	even	even

(d) **(i)** if f (x) + f (– x) = 0 ⇒ f is odd function.

(ii) if f (x) – f (– x) = 0 ⇒ f is odd function.

(e) The derivative of an odd function is an even function and derivative of an even function is an odd function.

(f) The square of even or an odd Function is always an even Function.

(g) Any function y = f (x) can written as y = f (x) = [odd part of f (x)] + [even part of f (x)]

i.e. $\quad y=f(x)=\left[\dfrac{f(x)-f(-x)}{2}\right]+\left[\dfrac{f(x)+f(-x)}{2}\right]$

Illustration 18

If $f(t) = \frac{t}{e^t - 1} + \frac{t}{2} + 1$**, show that** $f(t)$ **is an even function.**

Solution :

Since $$f(t) = \frac{1}{e^t - 1} + \frac{t}{2} + 1 \quad \text{... (i)}$$

Now, $$f(-t) = \frac{-t}{e^{-t} - 1} - \frac{t}{2} + 1$$

$$= \frac{te^t}{e^t - 1} - \frac{t}{2} + 1 \quad \text{... (ii)}$$

Subtracting (ii) from (i), we get

$$f(t) - f(-t) = \frac{t}{(e^t - 1)}(1 - e^t) + t$$

$$= -t + t = 0$$

$\therefore$ $f(-t) - f(t) = 0$. Hence $f(t)$ is an even function.

Illustration 19

Find out whether the given function is even, odd or neither even nor odd.

where, $$f(x) = \begin{cases} x|x| & , \quad x \le -1 \\ [1+x] + [1-x] & , \quad -1 < x < 1 \\ -x|x| & , \quad x \ge 1 \end{cases}$$

where || **and [ ] represents modulus & greatest integer function.**

Solution : The given function can be written as :

$$f(x) = \begin{cases} -x^2 & , \quad x \le -1 \\ 2 + [x] + [-x] & , \quad -1 < x < 1 \\ -x^2 & , \quad x \ge 1 \end{cases}$$

$$f(x) = \begin{cases} -x^2 & , \quad x \le -1 \\ 2 - 1 + 0 & , \quad -1 < x < 0 \\ 2 & , \quad x = 0 \\ 2 + 0 - 1 & , \quad 0 < x < 1 \\ -x^2 & , \quad x \ge 1 \end{cases}$$

$$f(x) = \begin{cases} -x^2 & , \quad x \le -1 \\ 1 & , \quad -1 < x < 0 \\ 2 & , \quad x = 0 \\ 1 & , \quad 0 < x < 1 \\ x^2 & , \quad x \ge 1 \end{cases}$$

which is clearly even as if f (– x) = f (x)

Thus, f (x) is even.

Illustration 20

Find out whether the given function is even or odd function, where

$$f(x) = \frac{x(\sin x + \tan x)}{\left[\frac{x+\pi}{\pi}\right] - \frac{1}{2}}, \text{ where } x \neq n\pi$$

[ ] denotes greatest integer function.

Solution :

$$f(x) = \frac{x(\sin x + \tan x)}{\left[\frac{x+\pi}{\pi}\right] \frac{-1}{2}} = \frac{x(\sin x + \tan x)}{\left[\frac{x}{\pi}\right] + 1 \frac{-1}{2}}$$

$$f(x) = \frac{x(\sin x + \tan x)}{\left[\frac{x}{\pi}\right] + 0.5}$$

$$f(-x) = \frac{-x(\sin(-x) + \tan(-x))}{\left[\frac{-x}{\pi}\right] + 0.5}$$

$$f(-x) = \begin{cases} \dfrac{x(\sin x + \tan x)}{-1 - \left[\frac{x}{\pi}\right] + 0.5} & , \quad x \neq n\pi \\ 0 & , \quad x = n\pi \end{cases}$$

$$f(-x) = -\left(\frac{x(\sin x + \tan x)}{\left[\frac{x}{\pi}\right] + \frac{1}{2}}\right)$$

f(– x) = – f(x)

$\therefore$ It is an odd function (if $x \neq n\pi$)

Identical function

Two functions f and g are identical if

(i) domain f = domain g

(ii) $f(n) = g(n)\ \forall\, n \in$ domain f or domain g.

Illustration 21

Find for what values of x the following functions are identical

(i) $f(x) = x$ **and** $g(x) = \sqrt{x^2}$

(ii) $f(x) = \dfrac{x^2}{x},\ g(x) = x$

Solution :

(i) Domain of g = $]-\infty, \infty[$

Domain of f = $]-\infty, \infty[$

$g(x) = \sqrt{x^2}$ = positive square root of $x^2 = |x| = x$, if $x \geq 0$

and $f(x) = x$

$\therefore$ f (x) and g (x) are identical $\forall\, x \in [0, \infty[$

(ii) Domain of f = $R - \{0\} =]-\infty, 0[\cup]0, \infty[$

Domain of g = $]-\infty, \infty[= R$

$f(x) = \dfrac{x^2}{x} = x$, when $x \neq 0$

$\therefore$ f (x) and g (x) are identical $\forall\ x \neq 0$

Explicit Function

If x and y are two variables connected by a relation such that y is expressed explicitly in terms of x or x is expressed explicitly in terms of y, i.e., y = f(x) or f(x) = y. Such functions are known as explicit functions.

For examples $y = x + 2$, $xy + y - 5 = 0$, $x^2 + y^2 = 5$ are explicit functions.

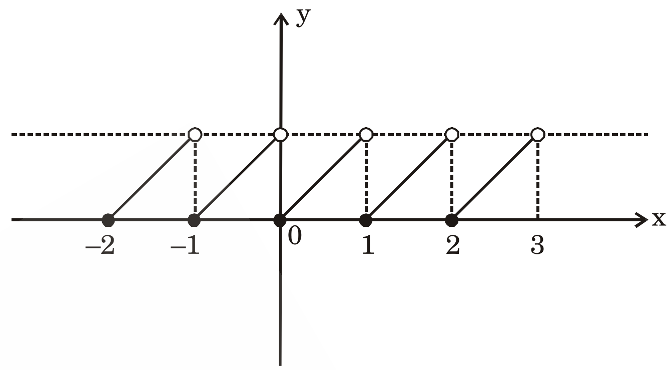
Implicit Function

If the variables x and y are connected by a relation such that neither y is expressed explicitly as a function of x nor x is expressed explicitly as a function of y. Such functions are known as implicit functions. These functions are expressed in the form

$f(x, y) = 0$

For example, $x^3 + y^3 + 3axy = 0$, $\tan(x^2 + y^2) + \cos(x + y) = e^x$ are implicit functions, of x, y.

PERIODIC FUNCTION



A function $f : D \rightarrow R$ is said to be a periodic function if there exists a positive real number p such that

$f(x + p) = f(x)$ for all $x \in D$. The least of all such positive numbers p is called the principal period of f. In general, the principal period is called the period of the function e.g. sin x and cos x are periodic functions, each having period 2π.

$\{x\} = x - [x]$ is a periodic function, the period being 1. The graph of $x - [x]$ is as shown in the figure.

Rules for finding the period of the periodic functions :

(i) If f (x) is periodic with period p, then $a f(x) \pm b$, where $a, b \in R$ $(a \neq 0)$ is also a periodic function with period p.

(ii) If f (x) is periodic with period p, then $f(ax \pm b)$, where $a \in R - \{0\}$ and $b \in R$, is also periodic with period $\frac{p}{|a|}$.

(iii) Let us suppose that f (x) is periodic with period p and g (x) is periodic with period q. Let r be the LCM of p and q, if it exists.

(a) If f (x) and g (x) cannot be interchanged by adding a least positive number k, then r is the period of f (x) + g(x).

(b) f (x) and g (x) can be interchanged by adding a least positive number k and if k < r, then k is the period of f (x) + g (x). Otherwise r is the period.

(iv) If f (x), g (x) are periodic functions with periods T_1, T_2 respectively then; we have,

$h(x) = a f(x) \pm b g(x)$ has period as,

$$\begin{cases} \text{LCM of } \{T_1, T_2\} & ; \text{ if } h(x) \text{ is not an even function} \\ \frac{1}{2}\text{LCM of } \{T_1, T_2\} & ; \text{ if } h(x) \text{ is an even function} \end{cases}$$

Note : **(1)** LCM of $\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right) = \frac{\text{LCM of } (a, c, e)}{\text{HCF of } (b, d, f)}$

(2) LCM of rational with rational is possible

LCM of irrational with irrational is possible

But LCM of rational and irrational is not possible.

Following results may be directly used

(i) sin x, cos x, sec x and cosec x are periodic functions with period 2π.

(ii) tan x and cot x are periodic functions with period π.

(iii) $|\sin x|$, $|\cos x|$, $|\tan x|$, $|\cot x|$, $|\sec x|$, $|\text{cosec } x|$ are periodic functions with period π.

(iv) $\sin^n x$, $\cos^n x$, $\sec^n x$ and $\text{cosec}^n x$ are periodic functions with period 2π and π according as n is odd or even respectively.

(v) $\tan^n x$ and $\cot^n x$ are periodic functions with period π, whether n is odd or even.

Illustration 22

Find the period of the function $f(x) = \frac{1}{2}\left\{\frac{|\sin x|}{\cos x} + \frac{|\cos x|}{\sin x}\right\}$

Solution : Period of $|\sin x|$, $|\cos x| = \pi$

Period of sin x, cos x = 2π

Period of $\frac{|\sin x|}{\cos x}$ = L.C.M. of π and $2\pi = 2\pi$

Period of $\frac{|\cos x|}{\sin x}$ = L.C.M. of π and $2\pi = 2\pi$

Period of $\frac{1}{2}\left\{\frac{|\sin x|}{\cos x} + \frac{|\cos x|}{\sin x}\right\}$ = L.C.M. 2π and $2\pi = 2\pi$

Illustration 23

Find the period of the following functions :

(i) $f(x) = \sin x + \{x\}$

(ii) $f(x) = \tan\frac{x}{3} + \sin 2x$

(iii) $f(x) = |\sin x| + |\cos x|$

Solution :

(i) Here f (x) = sin x + {x}

Period of sin x is 2π and that of {x} is 1. But the L.C.M. of 2π and 1 does not exist. Hence sin x + {x} is not periodic.

(ii) Here f (x) = tan x/3 + sin 2x. Here tan (x/3) is periodic with period 3π and and sin 2x is periodic with period π. Hence f (x) will be periodic with period 3π.

(iii) Here $f(x) = |\sin x| + |\cos x|$.

Now, $|\sin x| = \sqrt{\sin^2 x} = \sqrt{\dfrac{1-\cos 2x}{2}}$, which is periodic with period π.

Similarly, $|\cos x|$ is periodic with period π.

Hence, according to rule of LCM, period of $f(x)$ must be π.

But $\left|\sin\left(\dfrac{\pi}{2}+x\right)\right| = |\cos x|$ and $\left|\cos\left(\dfrac{\pi}{2}+x\right)\right| = |\sin x|$ [see rule (3) part (b)]

Since $\pi/2 < \pi$, period of $f(x)$ is $\pi/2$.

Illustration 24

Which of the following functions are periodic ? Also find the period if function is periodic.

(i) $f(x) = 10\sin 3x$ **(ii)** $f(x) = a\sin\lambda x + b\cos\lambda x$ **(iii)** $f(x) = \sin 3x$

(iv) $f(x) = \cos x^2$ **(v)** $f(x) = \sin\sqrt{x}$ **(vi)** $f(x) = \sqrt{\tan x}$

(vii) $f(x) = x - [x]$ **(viii)** $f(x) = x\cos x$

where x **is integral part of** x

Solution :

(i) $f(x) = 10\sin 3x$

Let $f(T + x) = f(x)$

$\Rightarrow$ $10\sin\{3T + 3x)\} = 10\sin 3x \Rightarrow \sin(3T + 3x) = \sin 3x$

$\Rightarrow$ $3T + 3x = n\pi + (-1)^n 3x$, where $n = 0, \pm 1, \pm 2, \ldots\ldots$

The positive values of T independent of x are given by

$3T = n\pi$, where $n = 2, 4, 6, \ldots\ldots$

$\therefore$ least positive value of $T = \dfrac{2\pi}{3}$

Hence $f(x)$ is a periodic function with period $\dfrac{2\pi}{3}$

(ii) $f(x) = a\sin\lambda x + b\cos\lambda x$

$$= \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\sin\lambda x + \frac{b}{\sqrt{a^2+b^2}}\cos\lambda x\right)$$

$$= \sqrt{a^2+b^2}\,(\cos\alpha\sin\lambda x + \sin\alpha\cos\lambda x), \text{ where } \tan\alpha = \frac{b}{a}$$

$= \sqrt{a^2 + b^2}\,\sin(\lambda x + \alpha)$

Which is a periodic function of period $\dfrac{2\pi}{|\lambda|}$

(iii) $f(x) = \sin^3 x = \dfrac{3\sin x - \sin 3x}{4} = \dfrac{3}{4}\sin - \dfrac{1}{4}\sin 3x$

Sin x is a periodic function of period 2 π and sin 3x is a periodic function of period $\dfrac{2\pi}{3}$.

Now L.C.M. of $\dfrac{2\pi}{1}$ and $\dfrac{2\pi}{3} = \dfrac{\text{L.C.M. of } 2\pi \text{ and } 2\pi}{\text{H.C.F. of 1 and 3}}$

$= \dfrac{2\pi}{1} = 2\pi$

Hence f (x) is a periodic function of 2π.

(iv) f (x) $= \cos x^2$

Let f (T + x) = f (x) $\Rightarrow \cos (T + x)^2 = \cos x^2$

$\Rightarrow$ $(T + x)^2 = 2n\pi \pm x^2$

From this no positive value of T independent of x is possible because x^2 on R.H.S. can be cancelled out only when T = 0

$\therefore$ f (x) is a non periodic function.

(v) f (x) = $\sin\sqrt{x}$

Let f (T + x) = f (x) $\Rightarrow \sin\sqrt{T + x} = \sin\sqrt{x}$

$\Rightarrow \sqrt{T + x} = n\pi + (-1)^n\sqrt{x}$

This will give no positive value of T independent of x because $\sqrt{x}$ on R.H.S. can be cancelled out only when T = 0.

$\therefore$ f (x) is a non periodic function.

(vi) $f(x) = \sqrt{\tan x}$

Let (T + x) = f(x) $\Rightarrow \sqrt{\tan(T + x)} = \sqrt{\tan x}$

$\Rightarrow$ tan (T + x) = tan x

$\Rightarrow$ T + x = $n\pi$ + x, = 0, $\pm$ 1, $\pm$ 2, ...

From this positive values of T independent of x are given by

$T = n\pi$, n = 1, 2, 3

$\therefore$ least positive value of T independent of x is π.

$\therefore$ f (x) is a periodic function of period π.

(vii) $f(x) = x - [x]$, where [x] is the integral part of x.

Let $f(T + x) = f(x)$

$\Rightarrow \quad (T + x) - [T + x] = x - [x]$

$\Rightarrow \quad T = [T + x] - [x]$ = an integer

Hence, least positive value of T independent of x is 1.

Hence f (x) is a periodic function of period 1.

(viii) $f(x) = x\cos x$

Let $f(T + x) = f(x) \Rightarrow (T + x)\cos(T + x) = x\cos x$

$\Rightarrow \quad T\cos(T + x) = x[\cos x - \cos(T + x)]$

From this no value of T independent of x is possible becase on R.H.S. one factor is x which is an algebraic function and on L.H.S. there is no algebraic function in x and hence x cannot be cancelled out.

Hence f (x) is a non periodic function.

Illustration 25

Let $f(x, y)$ be a periodic function satisfying $f(x, y) = f(2x + 2y, 2y - 2x)$ for all x, y. Let $g(x) = f(2^x, 0)$. Show that $g(x)$ is a periodic function with period 12.

Solution : Given, $f(x, y) = f(2x + 2y, 2y - 2x)$ ... (1)

$$\begin{aligned}\therefore \quad f(x, y) &= f(2x + 2y, 2y - 2x)\\ &= f[2(2x + 2y) + 2(2y - 2x), 2(2y - 2x) - 2(2x + 2y)] \quad \left[\begin{array}{r}\text{putting } x = 2x + 2y\\ \& \ y = 2y - 2x\end{array}\right]\\ &= f(8y, -8x)\end{aligned}$$

$$\begin{aligned}\text{Thus } f(x, y) &= f(8y, -8x)\\ &= f(-64x, -64y)\\ &= f[(-64)(-64x), (-64)(-64y)]\\ &= f(2^{12}x, 2^{12}y)\end{aligned}$$

$$\therefore \quad f(x, 0) = f(2^{12}x, 0)$$

$$\begin{aligned}\text{Now, } g(x) &= f(2^x, 0)\\ &= f(2^{12}.2^x, 0)\\ &= f(2^{x+12}, 0)\\ &= g(x + 12)\end{aligned}$$

Hence g (x) is a periodic function with period 12.

Illustration 26

Let f **be a real valued function defined for all real numbers** x **such that for some fixed** a **> 0,** $f(x+a)=\frac{1}{2}+\sqrt{f(x)-(f(x))^2}$ **for all real** x**. Show that** f (x) **is a periodic function. Also find its period.**

Solution : Given, $f(x+a)=\frac{1}{2}+\sqrt{f(x)-(f(x))^2}\ \forall\ x\in R$ ... (1)

$$\therefore \quad \left\{f(x+a)-\frac{1}{2}\right\}^2 = f(x)-(f(x))^2$$

$$= -\left(f(x)-\frac{1}{2}\right)^2+\frac{1}{4}$$

$$\Rightarrow \quad \left\{f(x+a)-\frac{1}{2}\right\}^2+\left\{f(x)-\frac{1}{2}\right\}^2=\frac{1}{4} \quad \text{... (2)}$$

$$\therefore \quad \left\{f(x+2a)-\frac{1}{2}\right\}^2+\left\{f(x+a)-\frac{1}{2}\right\}^2=\frac{1}{4} \quad \text{... (3)}$$

(3) − (2) $\Rightarrow \left\{f(x+2a)-\frac{1}{2}\right\}^2-\left\{f(x)-\frac{1}{2}\right\}^2=0$

$$\Rightarrow \quad f(x+2a)-\frac{1}{2}=f(x)-\frac{1}{2}$$

[$\therefore$ from (1), $f(x+a)-\frac{1}{2}>0\ \forall\ x\in R$

$\therefore$ $f(x-a+a)-\frac{1}{2}>0$ or, $f(x)-\frac{1}{2}>0\ \forall\ x\in R$

$\Rightarrow$ $f(x+2a)=f(x)\ \forall\ x\in R$ and fixed $a>0$

Hence f (x) is a periodic function with period 2a.

Find the range of the following functions

TRANSFORMATIONS

Transformation 1 :

Drawing the graph of $y = f(x) \pm a$, from the graph of $y = f(x)$

(a) To draw the graph of $y = f(x) + a$,
shift the graph of $y = f(x)$, a units in upward direction.

(b) To draw the graph of $y = f(x) - a$
shift the graph of $y = f(x)$, units in downward direction.

Logic : The graph can be taken as
$y \pm a = f(x)$, so we are just changing the value of y here.

Illustration 27

Plot the following :

(a) $y = |x| - 2$ **(b)** $y = \sin^{-1} x - 1$

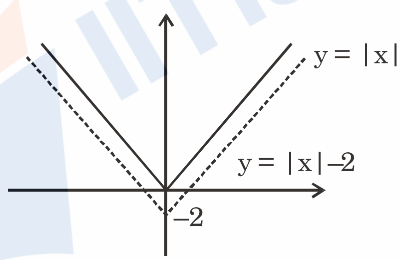
Solution :

(a) $y = |x| - 2$

we know the graph of $y = |x|$ (i.e. modulus function) and the given function can be written as $y + 2 = |x|$ also.

applying the transformation for $y = f(x) + a$

shift the curve of $y = |x|$ by 2 unit downward.



(b) $y = \sin^{-1}x - 1$

or we can write $y + 1 = \sin^{-1}x$

put $y + 1 \to y$

now, $y = \sin^{-1}x$

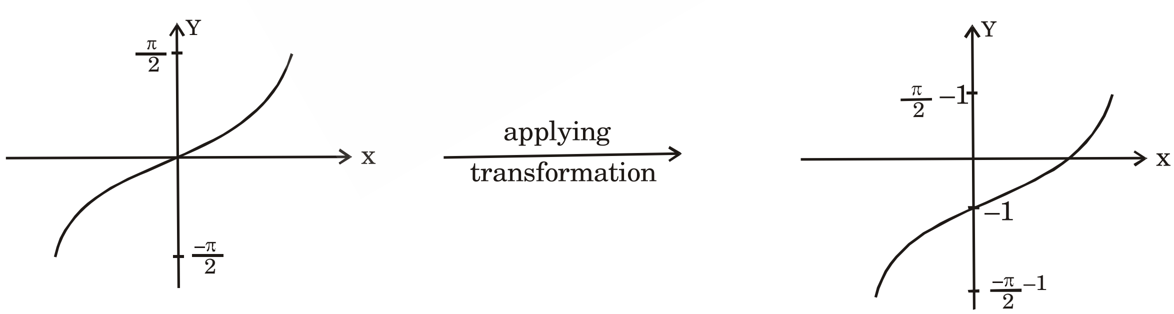


Illustration 28

Plot the following :

(a) $y = \sin x + 5$ **(b)** $y = \cos^2 x$

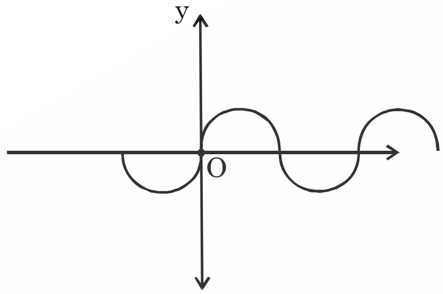
Solution :

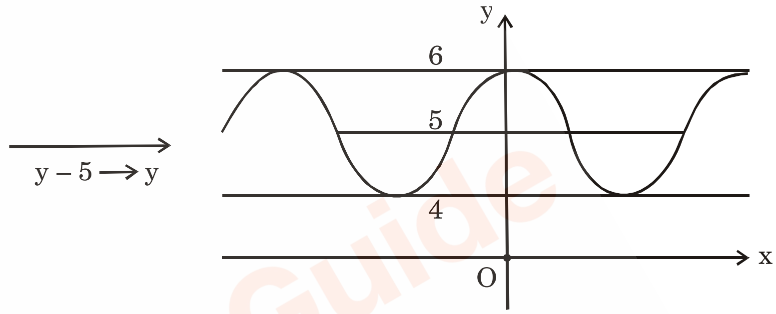
(a) $y = \sin x + 5$

writing it as $y - 5 = \sin x$

applying $y - 5 \rightarrow y$

$\Rightarrow$ $y = \sin x$





(b) $y = \cos^2 x$

we dont know the graph of $\cos^2 x$ but we do know the graph of cos x.

since $\cos 2x = 2\cos^2 x - 1$

$\Rightarrow$ $2\cos^2 x = \cos 2x + 1$

$\Rightarrow$ $\cos^2 x = \dfrac{\cos 2x + 1}{2}$

$\therefore$ $y = \dfrac{\cos 2x + 1}{2}$

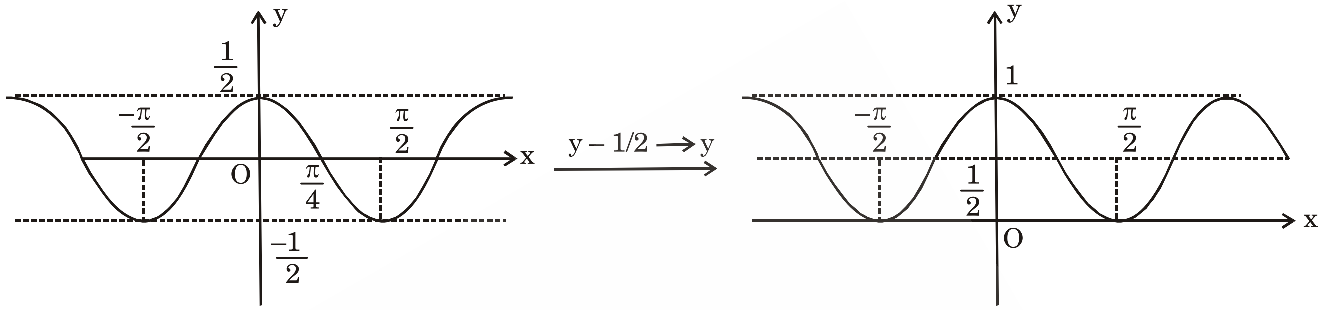
$\Rightarrow$ $y = \dfrac{1}{2} + \dfrac{\cos 2x}{2}$

$\Rightarrow$ $y - \dfrac{1}{2} = \dfrac{\cos 2x}{2}$

applying transformation $y - 1/_2 \rightarrow y$

$\Rightarrow$ $y = \dfrac{\cos 2x}{2}$

here $\frac{\cos 2x}{2}$ have half the period of cos x & also half of amplitude.



Transformation 2 :

To draw the graph of y = f (x ± a) from y = f (x)

(a) To draw y = f (x + a) from y = f (x) shift the graph of f (x) in 'a' units to left.

(b) To draw y = f (x – a) from y = f (x) shift the graph of f (x) by 'a' units to right.

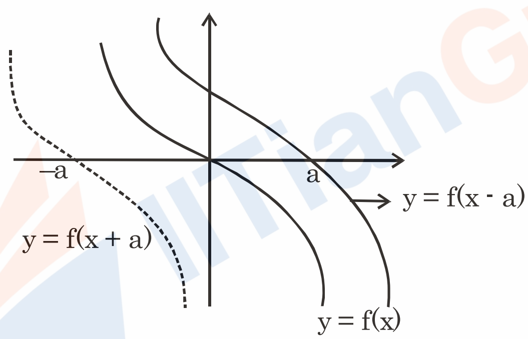


Illustration 29

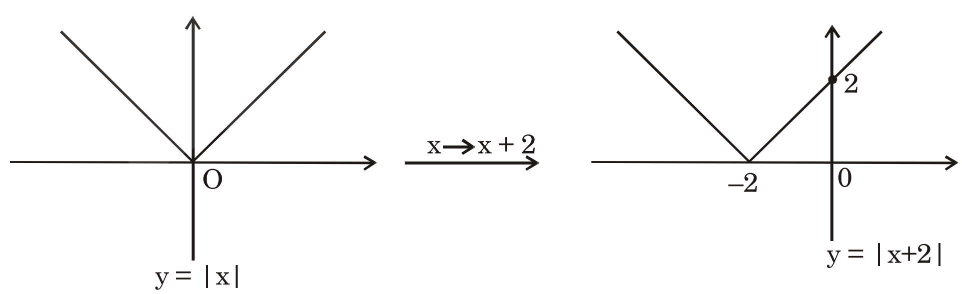
Plot

(a) $y = |x + 2|$ **(b)** $y = \sin\left(x - \frac{z}{4}\right)$ **(c)** $y = 4.2^x$

Solution : (a) for $y = |x + 2|$

by putting $x \to x + 2$

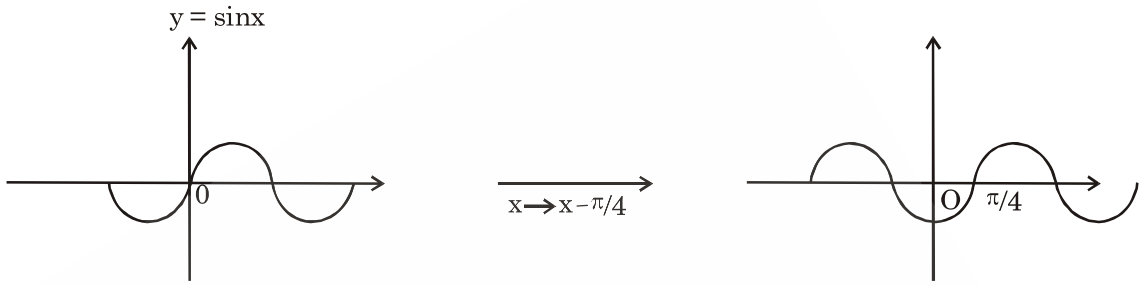
$y = |x|$



(b) $y = \sin\left(x - \frac{\pi}{4}\right)$

here we will draw the above graph from y = sin x

putting x for $x - \pi/_4$

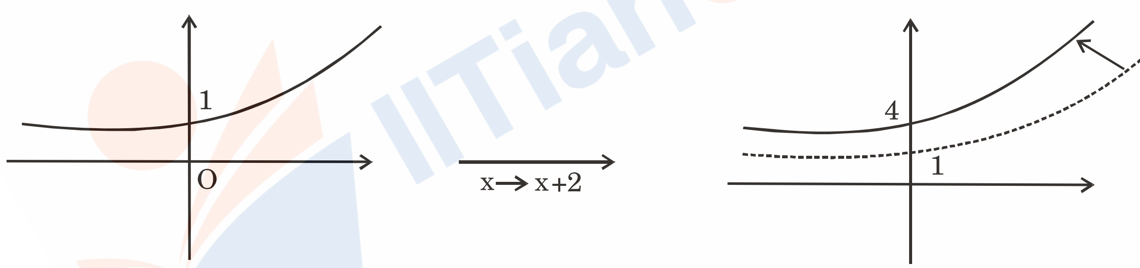


(c) $y = 4.2^x$

or we can write this as

$y = 2^2.2^x = 2^{x+2}$

we know the curve of 2^x, so



Transformation 3

To plot the curve of y = f (– x) from y = f (x)

1. Draw the graph y = f (x)
2. Then take the mirror image of y = f (x) in y-axis or we can say, turn the graph of f (x) by 180 about y-axis.

Illustration 30

Draw the graph of the following :

(a) $y = e^{-x}$ **(b)** y = **log (–** x**)**

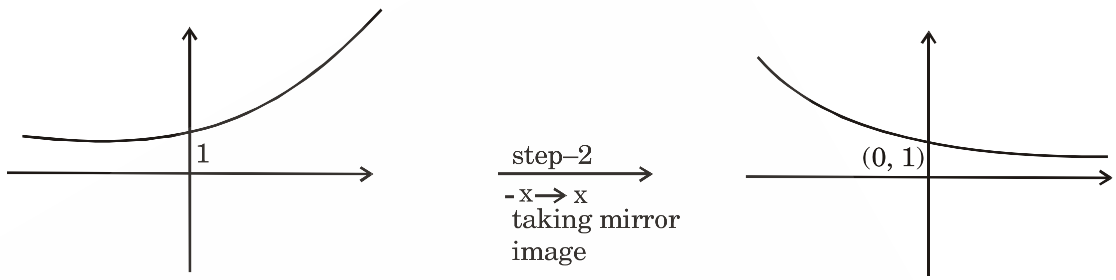
Solution :

(a) We will go stepwise for solving these curves.

Step 1 : Draw the graph of y = f (x)

so here putting – x as x

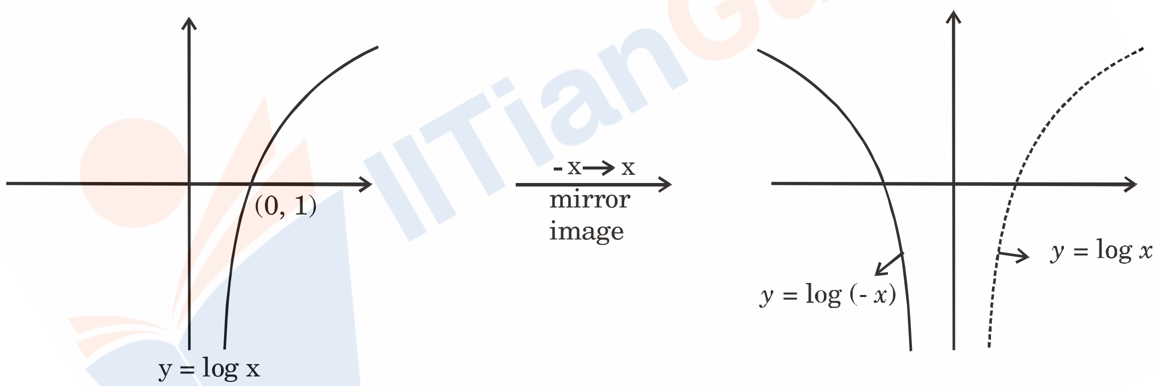
$\Rightarrow \quad y = e^x$



(b) y = log (– x)

– x → x

drawing the graph of y = log x



Transformation 4

To draw the graph of y = – f (x) from y = f (x)

Step 1 : Draw the graph of y = f (x)

Step 2 : Then take the mirror image of y = f (x) in x-axis.

Illustration 31

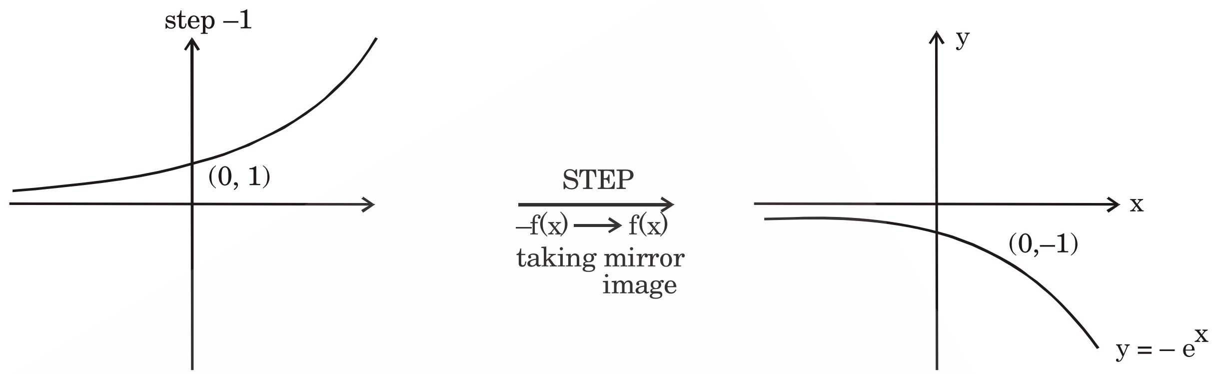
Plot the graph of the following curve :

(a) $y = -e^x$ **(b)** $y = \log\left(\frac{1}{x}\right)$

Solution :

(a) $y = -e^x$

we know the curve of $y = e^x$



(b) $y = \log\left(\frac{1}{x}\right)$

this can be written as $y = \log x^{-1}$

or $y = -\log x$ {by log properties}

now we have $y = -\log x$

Transformation 5

To plot $y = f(|x|)$ from $y = f(x)$

Step 1 : plot $y = f(x)$ curve

Step 2 : Remove the left portion of the graph

Step 3 : Take the reflection of right portion in y-axis (including right part also)

Illustration 32

Plot the curves of the following :

(a) $y = \log |x|$ **(b)** $y = x^2 - 2|x| + 3$

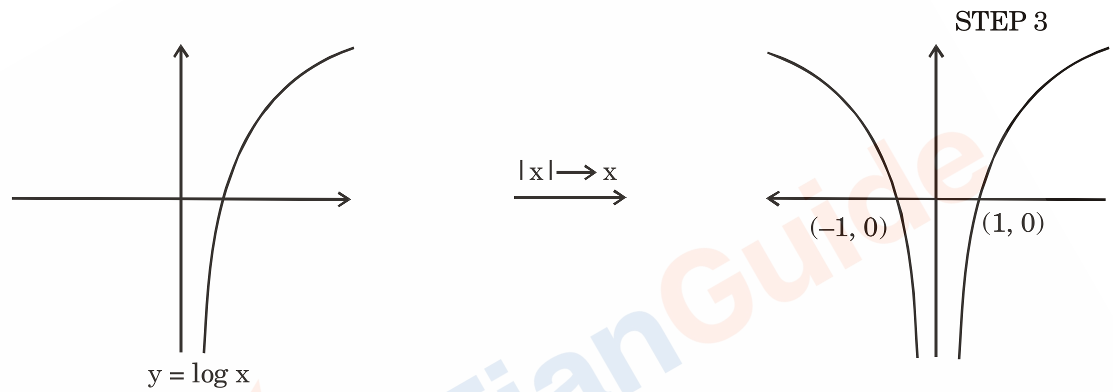
(c) $y = \dfrac{1}{|x|+1}$

Solution :

(a) given $y = \log |x|$

Step 1 : draw $y = f(x)$

Step 2 : remove left part already there is no left part here



(b) $y = x^2 - 2|x| + 3$

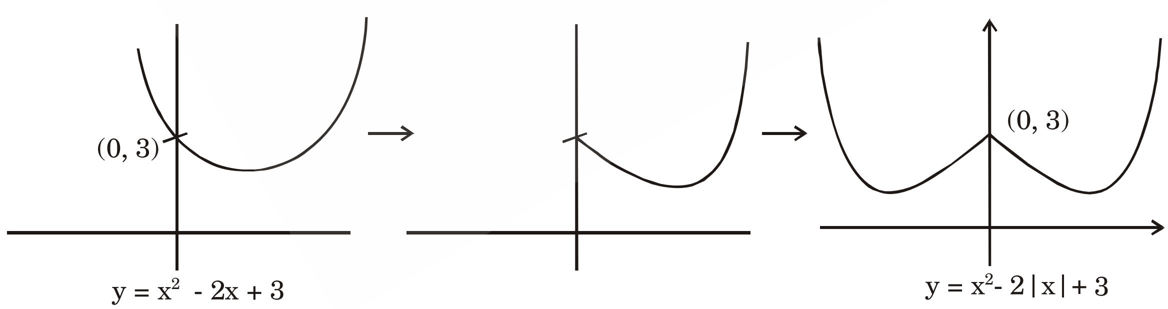
This is transformation of $|x| \to x$

Step 1 : putting $|x| \to x$

$\therefore\ y = x^2 = 2x + 3$ drawing the curve

Step 2 : Remove the left part

Step 3 : taking reflection



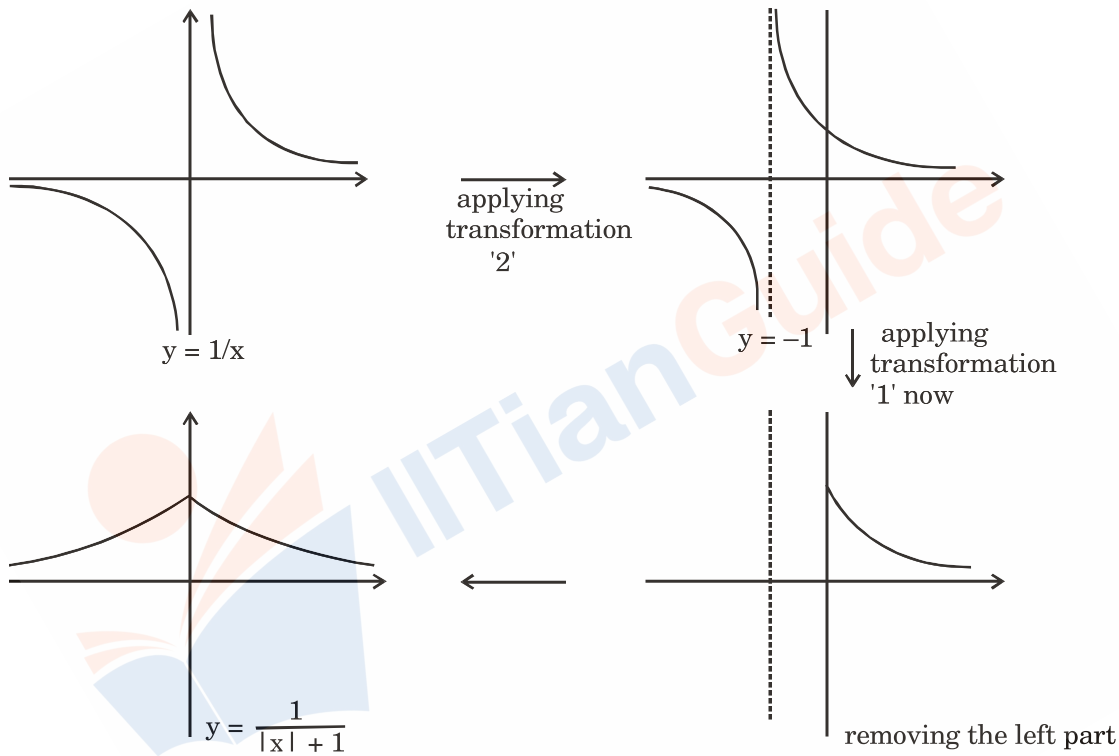
(c) $y = \dfrac{1}{|x| + 1}$

This one includes 2 transformations

1. $|x| \to x \quad \Rightarrow \quad y = \dfrac{1}{x+1}$

2. $x + 1 \to x \quad \Rightarrow \quad y = \dfrac{1}{x}$

So first drawing the curve $y = \dfrac{1}{x}$



Note : The order of applying the transformations is very important, otherwise we will get wrong answer.

Transformation 6

To plot $y = |f(x)|$ from the curve of $y = f(x)$

Step 1 : Draw $y = f(x)$

Step 2 : Take mirror image of portion below x-axis in x-axis. (removing the lower portion).

Illustration 33

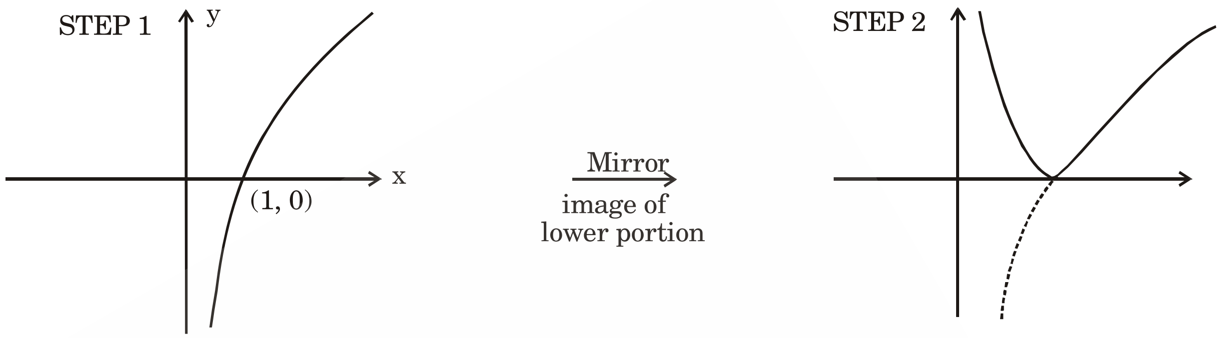
Plot the graph of the following curves

(a) $y = |\log x|$ **(b)** $y = |x^2 - 3x + 2|$

Solution :

(a) given y = |f(x)| = |log x|

where f (x) = log x

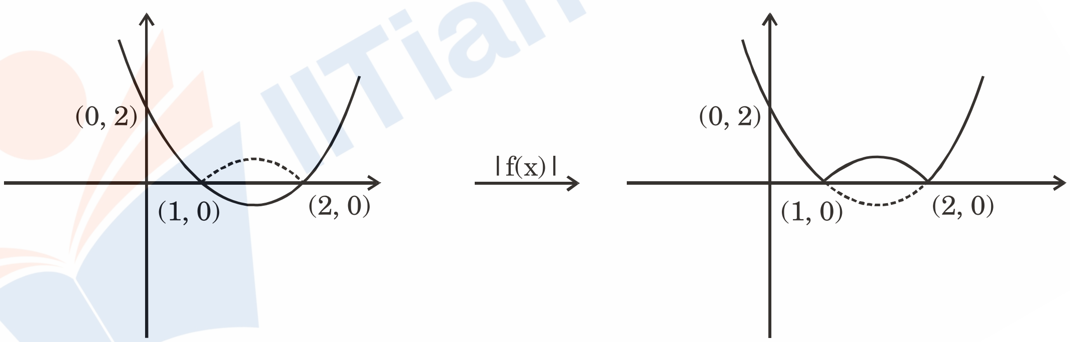


(b) $y = |x^2 - 3x + 2|$

$\Rightarrow \quad y = |(x - 1)(x - 2)$

applying transformation $|f(x)| \to f(x)$

$\therefore \quad f(x) = (x - 1)(x - 2)$



Transformation 7

To plot the graph of |y| = f (x) from y = f (x)

Step 1 : draw y = f (x)

Step 2 : Remove the lower portion i.e. below x-axis.

Step 3 : Take mirror image of upper part in lower part, keeping the upper part also.

Illustration 34

Draw the graph of the following :

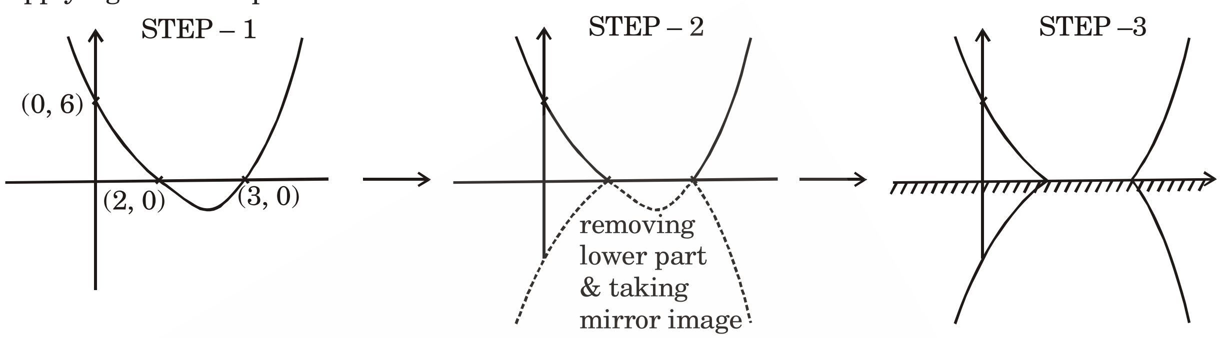
(a) **|y| = (x – 2) (x – 3)**

(b) **|y| = log** x

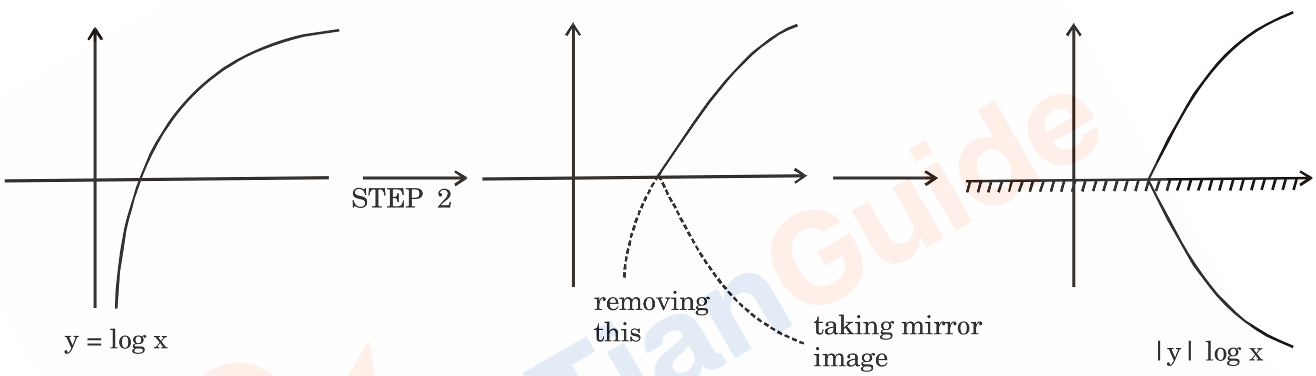
Solution :

(a) given $|y| = (x - 2)(x - 3)$

applying the 3 steps



(b) $|y| = \log x$



Transformation 8

To plot $x = f(y)$ from $y = f(x)$

Step 1 : Draw $y = f(x)$

Step 2 : Take reflection in line $y = x$, also called reflection about origin.

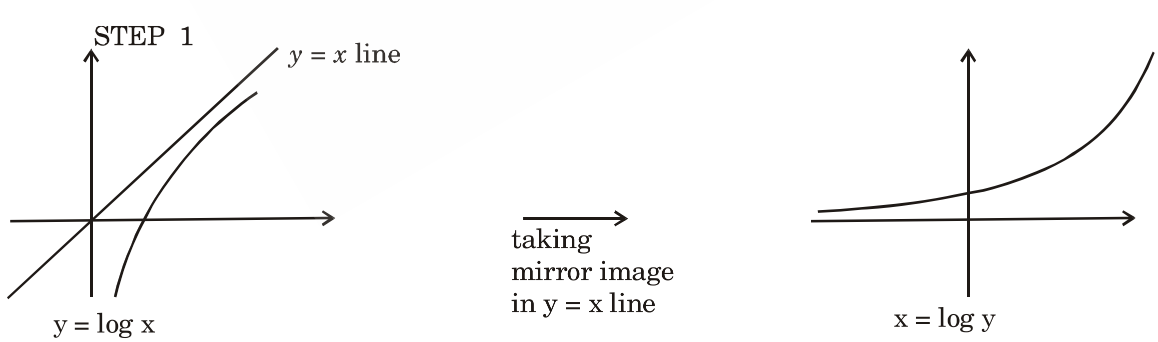
Illustration 35

Plot the graph of the following :

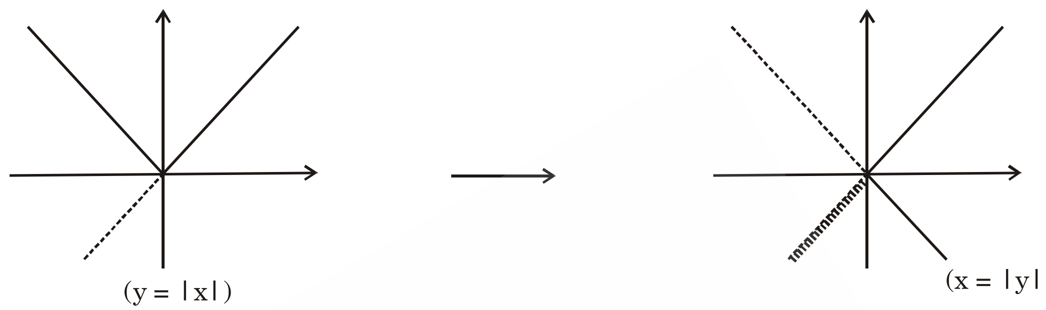
(a) $x = \log y$ **(b)** $x = |y|$

Solution :

(a) now here x & y are interchanged original function was $y = \log x$



(b) we have $x = |y|$ & we know the graph of $y = |x|$, which we get by replacing x by y & y by x.



Transformation 9

To plot $x = |f(y)|$ from the graph $x = f(y)$

Step 1 : Draw the graph of $x = f(y)$, using transformation 8.

Step 1 : Take reflection of Left portion in y axis.

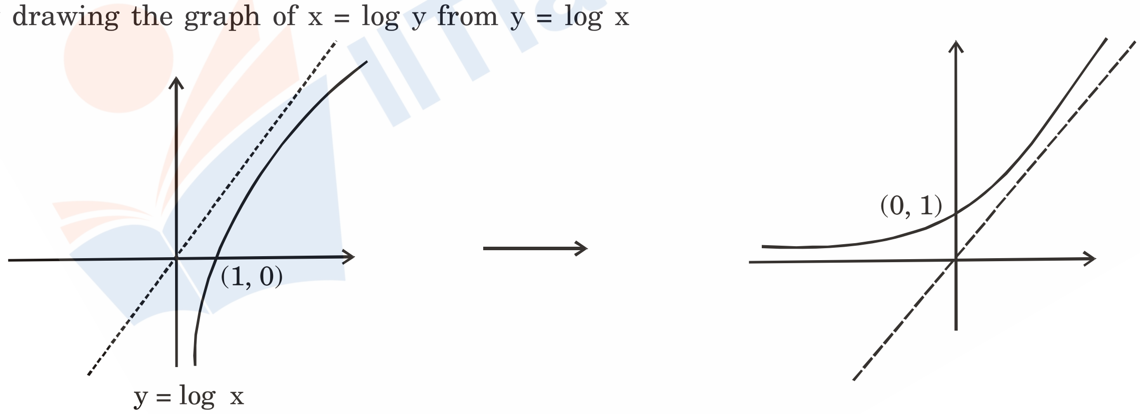
Illustration 36

Plot the graph of the following :

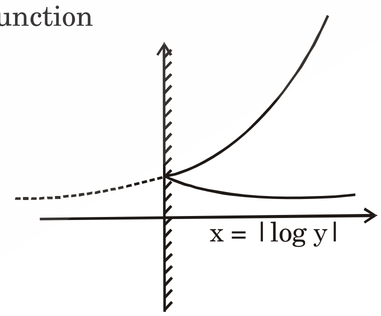
(a) $x = |\log y|$

Solution :

Firstly drawing the graph of $x = \log y$ from $y = \log x$



now applying step 2 for modulus function



Note : This transformation is not valid for $g(x) = |f(y)|$ i.e. on L.H.S. only x should be there & no other function.

Transformation 10

To plot the graph of y = [f (x)] from y = f (x)

Step 1 : Draw y = f (x)

Step 2 : Draw horizontal lines after every unit distance i.e. $y \in k$, where k integers.

Step 3 : From the point of intersection (as obtained from step 2), draw vertical lines.

Step 4 : From the intersection points draw horizontal lines upto the nearest vertical line (towards right). The line drawn should be below the curve for that region.

We will understand the steps with the help of examples.

Illustration 37

Draw the graph of following curves :

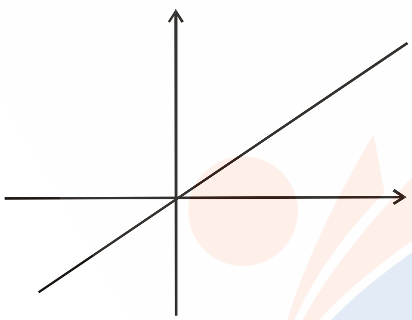
(i) y = [x] **(ii)** y = [x^3] **(iii)** y = [2sinx]

Solution :

(a) We will go stepwise, so as to understand the procedure.

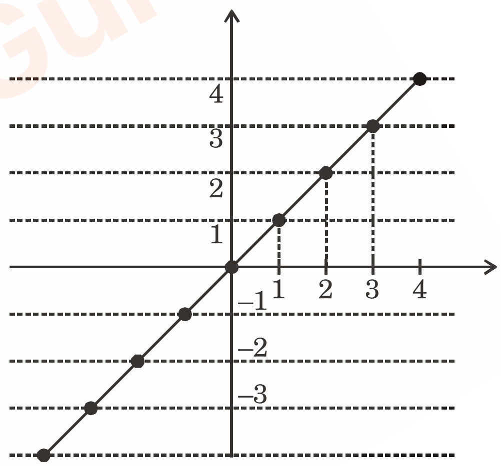
we have y = [x]

Step 1 : draw y = x

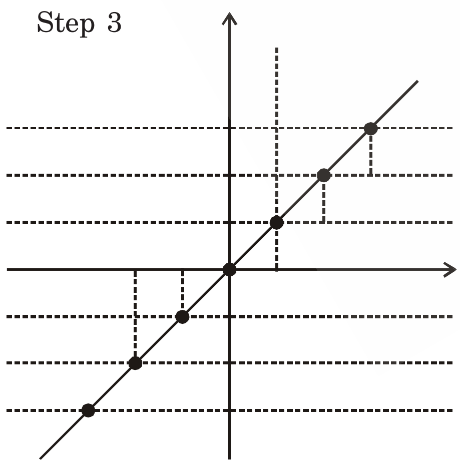


drawing horizontal lines →

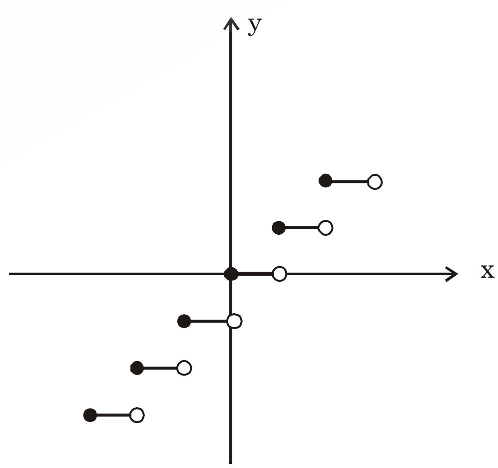
Step 2



Step 3



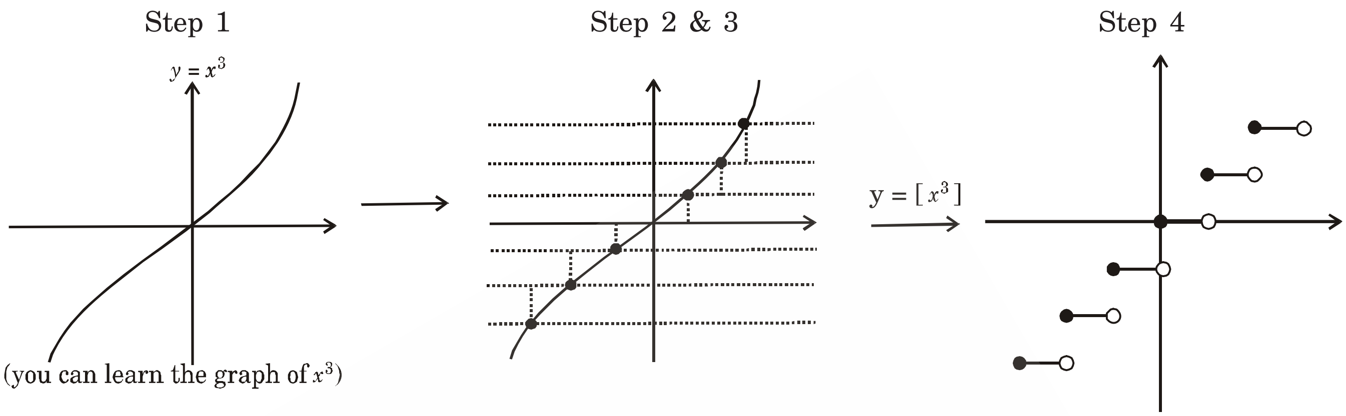
drawing final lines →



drawing vertical lines downward to x-axis

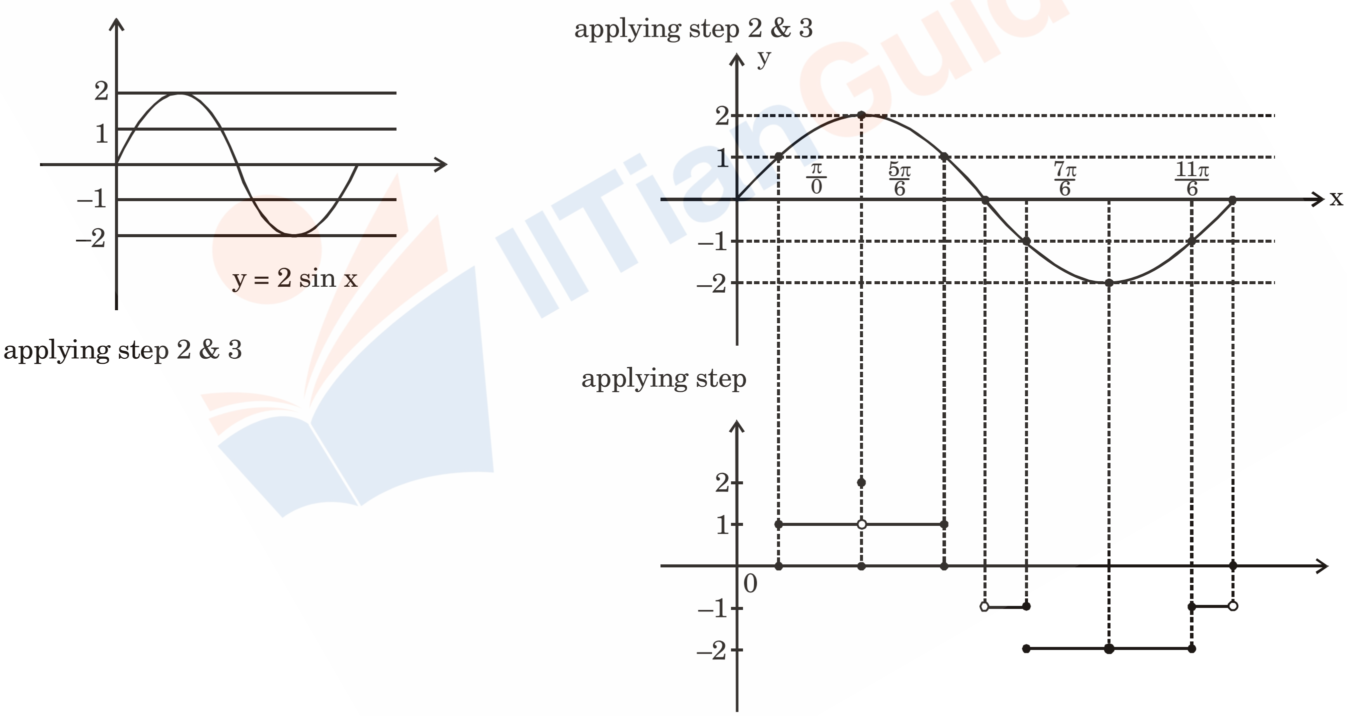
final graph of y = [x]

(b) $y = [x^3]$



(c) $y = [2 \sin x]$

drawing first 2 sin x, which is almost same curve as sin x but has an amplitude of 2 rather than 1.



Transformation 11

To plot $y = f([x])$ from the graph of $y = f(x)$

Step 1 : Draw $y = f(x)$

Step 2 : Draw vertical lines on every integral point of x i.e. $x = k$ where $k \in I$ (integers)

Step 3 : Draw horizontal lines from point of intersection to the nearest right vertical line.

Illustration 38

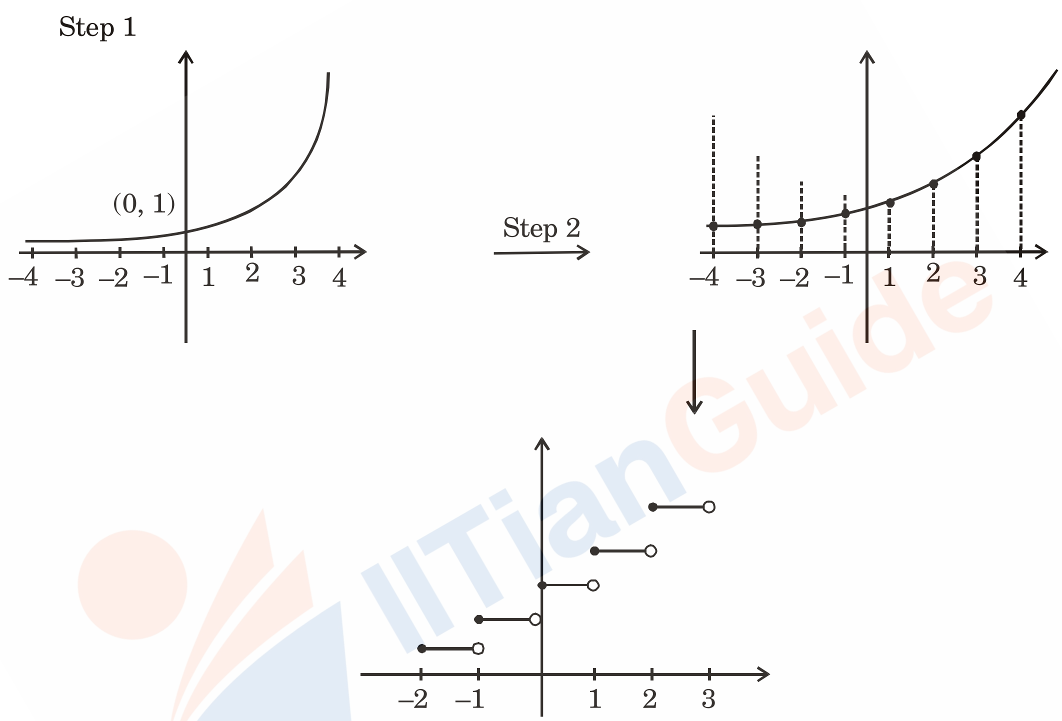
Plot the following curves :

(a) $y = e^{[x]}$ **(b)** $y = \sin [x]$

Solution :

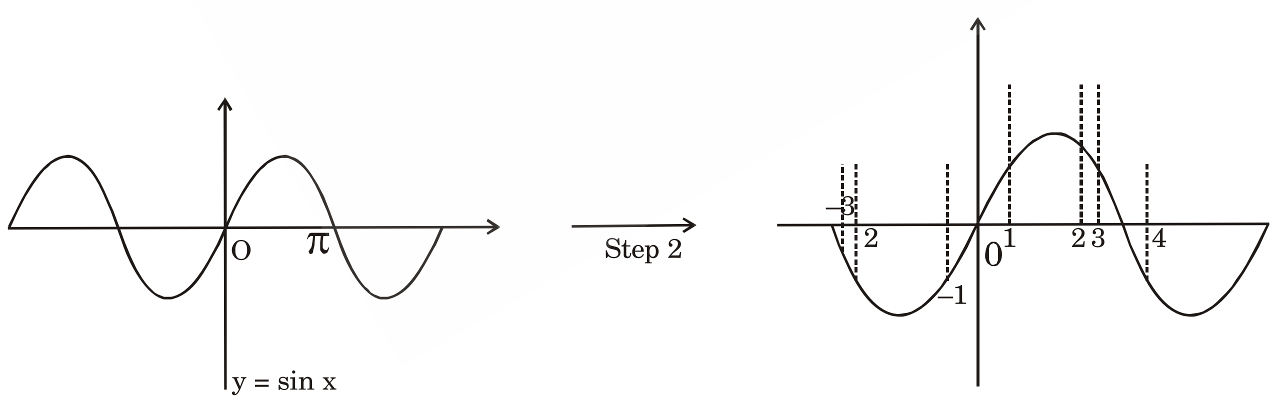
(a) given $y = e^{[x]}$

we know the graph of $y = e^x$



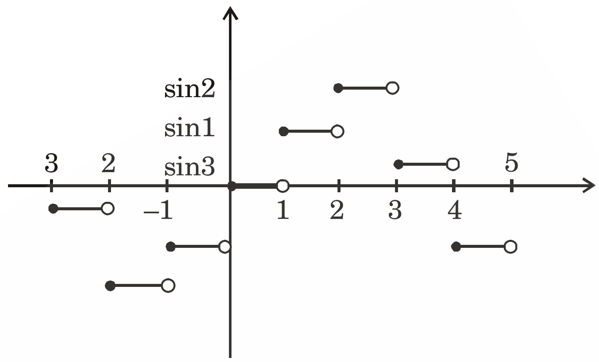
(b) $y = \sin [x]$

here we know the graph of $y = \sin x$



In step 2 we have marked lines for $x = k$ (where $k \in$ integers)

Note here that π has a value 3.14 (approx.), so now we can understand that we will get point $x = 1, 2, 3$ between 0 and π.



Note here that the figure in step 2 has parts above and below x-axis in (3, 4) but in final graph the graph between (3, 4) is above x axis.

This is so because, for $3 \le x < 4$, $[x] = 3$ only, so the value will be sin 3.

Also do not get confuse in values sin 1, sin 2 & sin 3.

for $1 \le x < 2$; $\sin [x] = \sin 1 \sim .84$

$2 \le x < 3$; $\sin [x] = \sin 2 \sim .909$

$3 \le x < 4$; $\sin [x] = \sin 3 \sim .14$

& 1, 2 & 3 are in radians.

Transformation 12

To plot $[y] = f(x)$, from $y = f(x)$

Step 1 : Draw $y = f(x)$

Step 2 : Draw horizontal lines at a unit distance i.e. $y = k$ (k belongs to set of integers)

Step 3 : Draw vertical lines from the point of intersection up till next upper horizontal line Include only the lower point.

Illustration 39

Plot the following graph :

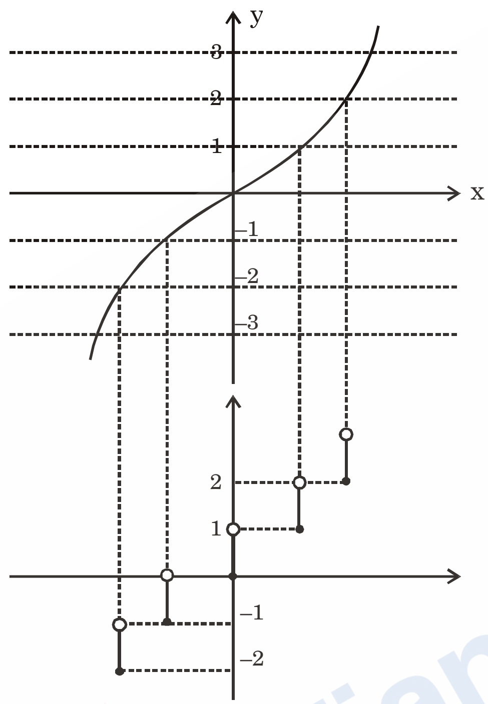
(a) $[y] = x^3$

(b) $[y] = x^2 - 2$

Solution :

(a) given [y] = x^3, we known the graph of $y = x^3$

Step 1 & 2



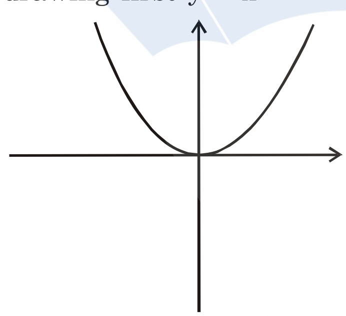
(b) $[y] = x^2 - 2$

This one includes two transformations :

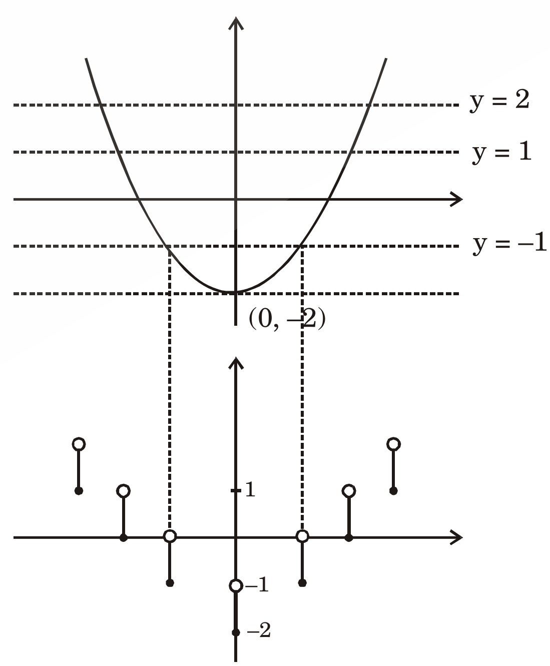
1. $[y] \rightarrow y \quad \Rightarrow \quad y = x^2 - 2$

2. $y + 2 \rightarrow 2 \Rightarrow \quad y = x^2$

drawing first $y = x^2$



applying transformation



So, you can note here that upper points on the vertical lines are not included.

Transformation 13

To plot x = [f (y)] from x = f (y)

Step 1 : draw x = f (y) by using transformation 8.

Step 2 : draw vertical lines at a unit distance
i.e. x = k (k ∈ integers)

Step 3 : draw vertical lines from the point of intersection uptil the above intersection point.

Illustration 40

Draw the graph of $x = \left[\sqrt{y}\right]$

Solution :

Given $x = \left[\sqrt{y}\right]$

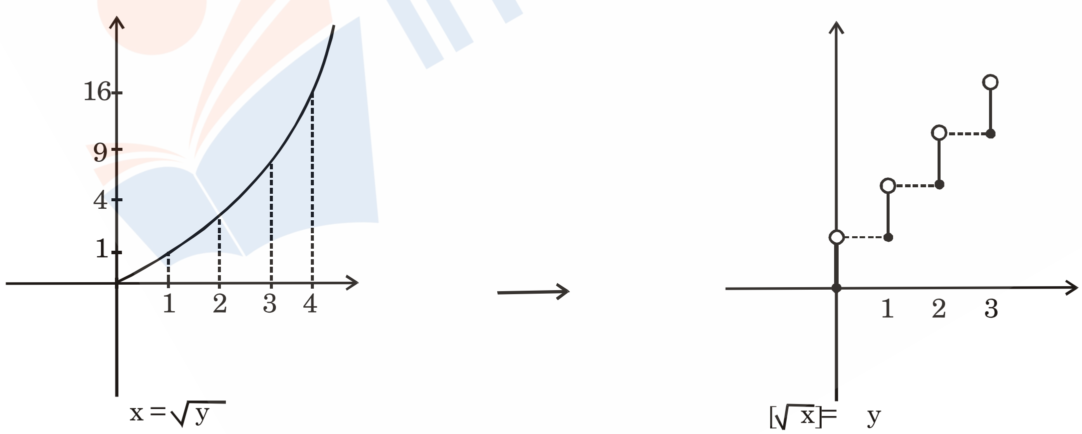
first we will draw the graph of $x = \sqrt{y}$

for $x = \sqrt{y}$, $y \geq 0$ (as it is a property of under root function)

squaring $\Rightarrow x \geq 0$

$\Rightarrow x^2 = y$

here we will draw in the region where $x \geq 0$ as stated by (1)



Note : This transformation is not valid for functions of form g (x) = [f (y)]

Transformation 14

Plot the graph of y = f ({x}) from y = f (x)

Step 1 : draw the graph of y = f(x) in the interval [0, 1]

Step 2 : Repeat the same graph as in step 1, with a period of 1.

Illustration 41

Plot the graph of the following :

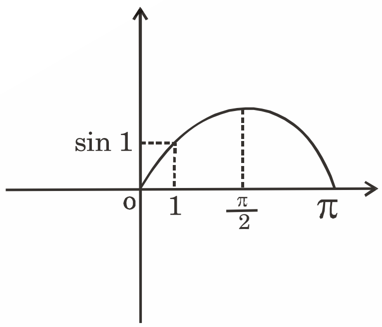
(a) $y = \sin(x - [x])$ **(b)** $y = \dfrac{2^x}{2^{[x]}}$

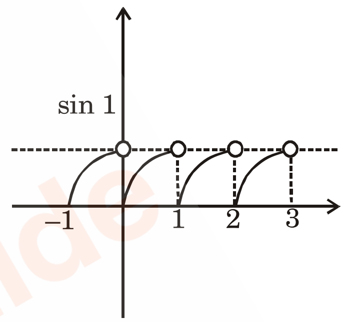
Solution :

(a) given $y = \sin(x - [x])$

& we know that $x - [x] = \{x\}$

$\therefore$ we have to draw $y = \sin x$ in $[0, 1]$



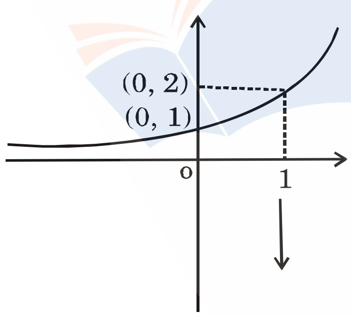


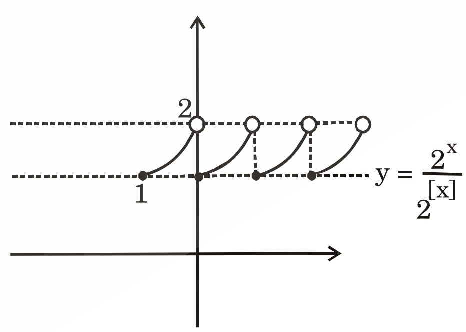
(b) $y = \dfrac{2^x}{2^{[x]}}$

this could be written as $y = 2^{x - [x]} \Rightarrow y = 2^{\{x\}}$

for this we will draw $y = a^x$ with $a > 0$

for $\{x\}$ graph we will only check for the output between 0 and 1.





Transformation 15

To draw the graph of $y = \{f(x)\}$ from $y = f(x)$

Step 1 : Draw the graph of $y = f(x)$

Step 2 : Transfer the graph between the interval

$y = 0$ & $y = 1$

Illustration 42

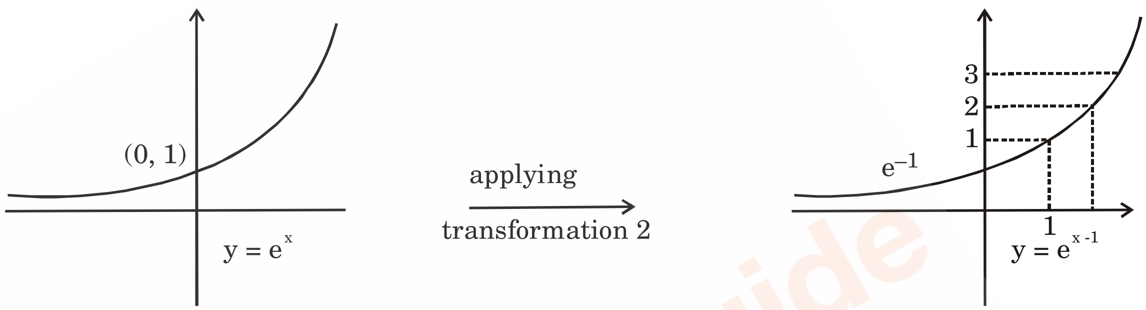
Plot the graph of the following :

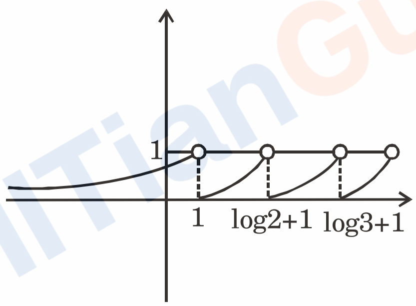
(a) $y = \left\{e^{x-1}\right\}$ **(b)** $y = \{2\sin x\}$

Solution :

(a) given that $y = \left\{e^{x-1}\right\}$

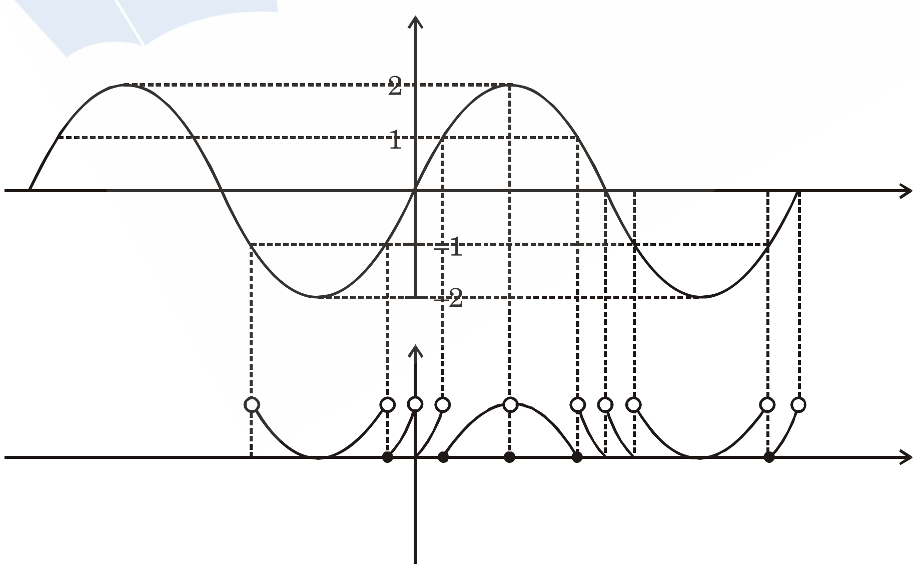
1. $\{f(x)\} \to f(x) \Rightarrow y = e^{x-1}$
2. $x - 1 \to x \Rightarrow y = e^{x}$





(b) $y = \{2\sin x\}$

We know the graph of $y = 2\sin x$, as we have done that earlier also.



Transformation 15

$y = f(x) \rightarrow \{y\} = f(x)$

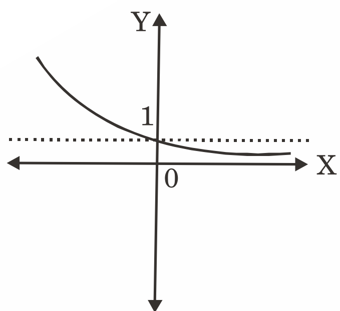
To draw $\{y\} = f(x)$. Draw the graph of $y = f(x)$ then retain the graph of $y = f(x)$ which lies between $y \in [0, 1)$ and neglect the graph for other values. Also repeat this graph in the same interval for x, but for all intervals $y \in [n, n + 1)$.

Illustration 43

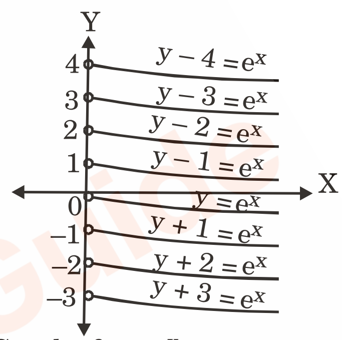
Plot the graph of $\{y\} = e^{-x}$

Solution :

(i) $\{y\} = e^{-x} \xleftarrow[y=\{y\}]{} y = e^{-x}$



Graph of $y = e^x$



Graph of $y = e^x$

Transformation 16

$$x = f(y) \longrightarrow x = \{f(y)\}$$

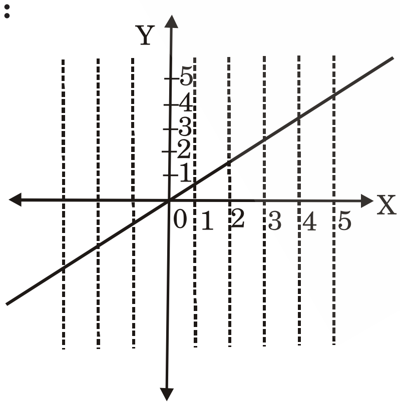
To draw $x = \{f(y)\}$. Draw $x = f(y)$. Draw vertical lines corresponding to integral values of x. Transfer the graph between two consecutive vertical lines to the region lying between $x = 0$ & $x = 1$. Don't include the points lying on $x = 1$.

Illustration 44

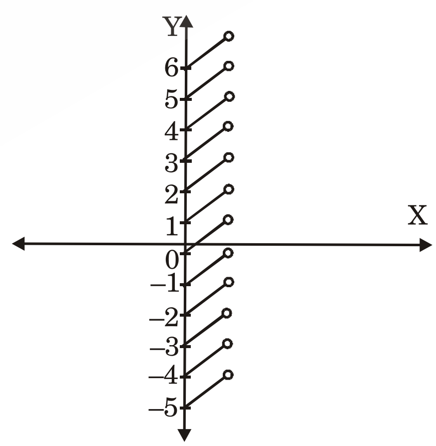
Plot (i) $x = \{2y\}$

Solution :

$$x = \{2y\} \xleftarrow[f(y) = f\{y\}]{} x = 2y$$



Graph of $x = 2y$



Graph of $x = \{2y\}$

Note : This transformation is not valid for drawing $g(x) = \{f(y)\}$ from $g(x) = f(y)$

Transformation 17

$$y = f(x) \longrightarrow y = \text{sgn}(f(x))$$

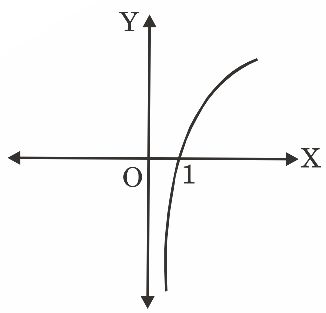
To Draw y = sgn (f (x)). Draw y = f (x). Then draw y = 1 for which f (x) > 0 and y = – 1 for which f (x) < 0 and y = 0 for which f (x) = 0.

Illustration 45

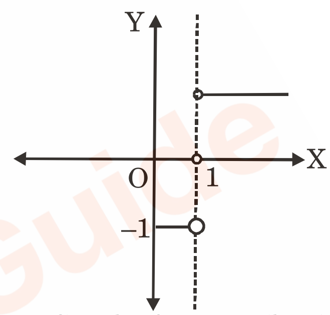
Plot the graph of y = sgn (log x)

Solution :

$$y = \text{sgn}(\log x) \xleftarrow[y=\{y\}]{} y = \log x$$



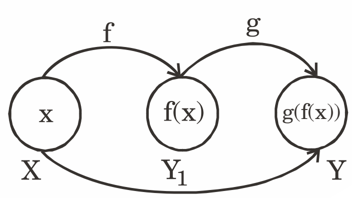
Graph of $y = \log_e x$



Graph of y = sgn (logx)

COMPOSITE FUNCTIONS

Let us consider two functions, $f : X \to Y_1$ and $g : Y_1 \to Y$. We define function $h : X \to Y$ such that h (x) = g(f(x)). To obtain h(x), we first take the f-image of an element $x \in X$ so that $f(x) \in Y_1$, which is the domain of g (x). Then take g-image of f(x), i.e. g(f(x) i.e. g (f(x)) which would be an element of Y. The adjacent figure clearly shows the steps to be taken.



The function 'h' defined above is called the composition of f and g and is denoted by gof.

Thus (gof)x = g(f(x). Clearly Domain (gof) = {x : x∈ Domain (f), f(x) ∈ Domain (g)} Similarly we can define, (fog)x = f (g(x)) and Domain (fog) = {x : x ∈ Domain (g), g (x) Domain (f)}. In general fog ≠ g of.

Explanation :

(i) To understand the concept of complete function consider fog (x) :

$$x \longrightarrow \underset{\text{Ist}}{\boxed{g}} \xrightarrow{g(x)} \underset{\text{IInd}}{\boxed{f}} \xrightarrow{fog(x)}$$

in the above diagram for Ist block 'x' is the independent variable and corresponding g(x) is the dependent variable. But for IIrd block f (x) i.e. the dependent variable of Ist block is independent variable of the IInd block and corresponding fog (x) is the dependent variable of IInd block.

(ii) fof(x) is $x \longrightarrow \underset{\text{Ist}}{\boxed{f}} \xrightarrow{f(x)} \underset{\text{IInd}}{\boxed{f}} \xrightarrow{fof(x)}$

(iii) gof(x) is $x \longrightarrow \underset{\text{Ist}}{\boxed{f}} \xrightarrow{f(x)} \underset{\text{IInd}}{\boxed{g}} \xrightarrow{gof(x)}$

(iv) gog(x) is $x \longrightarrow \underset{\text{Ist}}{\boxed{g}} \xrightarrow{g(x)} \underset{\text{IInd}}{\boxed{g}} \xrightarrow{gog(x)}$

General steps for determining composite functions

Step 1 : Find critical points

(a) Draw graph of first block.

(b) Draw y = k (horizontal lines)

$k \in$ critical pt(s) for second block

(c) Make pt(s) of intersection & find corresponding values of x.

(d) Critical pt(s) of first block & values obtained in c are critical pt(s) of composite function.

Step 2 : Divide interval about critical point.

Step 3 : In each and every interval find appropriate definition of the function.

Illustration 46

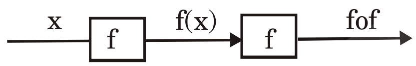
Consider the function as defined as under

$$f(x) = \begin{cases} \mathbf{1} + x & , \quad \mathbf{0} \le x \le \mathbf{2} \\ \mathbf{3} - x & ; \quad \mathbf{2} < x \le \mathbf{3} \end{cases}$$

Evaluate f **[**f (x)**]**

Solution :

Here we have to evaluate fof (x)

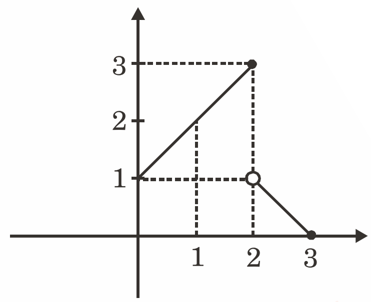


According to the rules that we have mentioned above,

Step 1 : It says draw the graph of first block.

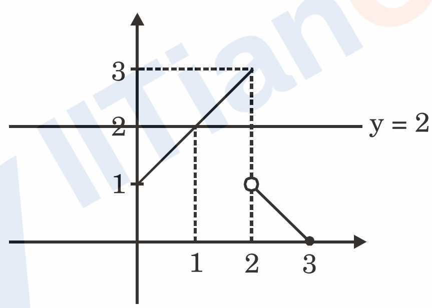
i.e. f (x)

we know $f(x)\begin{cases} 1+x & ; 0 \le x \le 2 \\ 3-x & ; 2 < x \le 3 \end{cases}$



(b) Now we have to y = k where k $\in$ critical points of the above function which is 2 in this question.

$\therefore$ draw y = 2 line & find the point of intersection, which comes out to be 1



(c) now the critical point found will be the critical point of the composite function. Also the critical point of first block will add to the critical points of composite function.

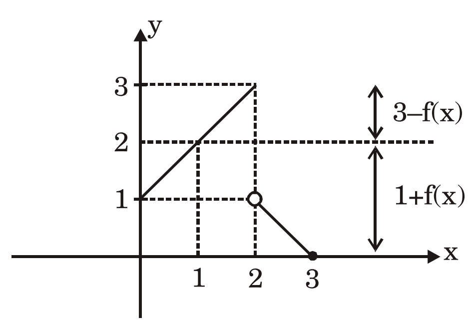
Step 2 : For composite function; now

the interval will be divided as follows

$0 \le f(x) \le 2$ &

$2 < f(x) \le 3$

& now seeing for values, for these values.



Step 3 : Now finding appropriate values for their intervals

$$f[f(x)] = \begin{cases} 1+f(x) & 0 \le x \le 1 \\ 3-f(x) & 1 < x \le 2 \\ 1+f(x) & 2 < x \le 3 \end{cases}$$

as can be seen in the above graph.
Now put values of f (x) corresponding to the interval of x.

$$\text{fof} = \begin{cases} 1+(1+x) & 0 \le x \le 1 \\ 3-(1+x) & 1 < x \le 2 \\ 1+(3-x) & 2 < x \le 3 \end{cases}$$

$$\Rightarrow \quad \text{fof} = \begin{cases} 2+x & 0 \le x \le 1 \\ 2-x & 1 < x \le 2 \\ 4-x & 2 < x \le 3 \end{cases}$$

Illustration 47

Evaluate and draw the graph of following functions :

(a) f (x) = **$\sin^{-1}$ (sin** x) **(b)** f (x) = **sin ($\sin^{-1}$** x)

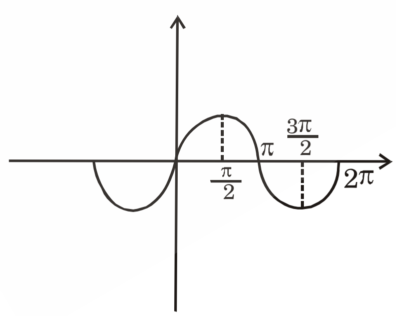
Solution :

(a) f (x) = $\sin^{-1}$ (sin x)

This can be considered a composite function with f (g (x)) as $\sin^{-1}$ (sin (x))

It is clear that $D_f \in R$, since sin x is valid for all values of x & it value ranges from – 1 to + 1 which satisfies $\sin^{-1}$ function also.

We know the graph of y = sin x from we get critical points as $\frac{\pi}{2}, \frac{3\pi}{2} \ldots (2n+1)\frac{\pi}{2}$ (We will cover critical points later also).



For now you can say that in such curves the points on which the tangent is parallel to x-axis are the critical points.

for $0 \le x \le \frac{\pi}{2}$ $\qquad$ $\sin^{-1}(\sin x) = x$

for $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$ $\qquad$ $\sin^{-1}(\sin x) = \pi - x$

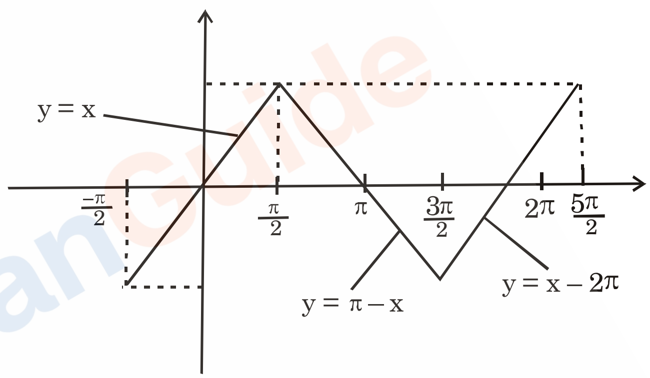
Now this is important to understand this point, whenever we use inverse functions we represent them in their principal value branch.

& $\sin^{-1} x$ principal value branch is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so for value between $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right]$ we have to convert them into principal values, which we do by subtracting it from π.

So similarly for interval $\frac{3\pi}{2} < x < \frac{5\pi}{2}$, we will subtract them by 2π and so on.

So the final graph becomes

$$f(x) = \sin^{-1}(\sin x) = \begin{cases} x & ; -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \pi - x & ; \frac{\pi}{2} < x \le \frac{3\pi}{2} \\ x - 2\pi & ; \frac{3\pi}{2} < x < \frac{5\pi}{2} \end{cases}$$



(b) $f(x) = \sin(\sin^{-1} x)$

first of all we will find the domain of the function for $\sin^{-1} x$, x can only take values between – 1 & 1

$\therefore \quad D_f \in [-1, 1]$

& since values between – 1 & 1 lie between $\frac{-\pi}{2}$ & $\frac{\pi}{2}$ (i.e. the principal value branch)

$\sin(\sin^{-1} x) = x$

$\therefore \quad f(x) = \sin(\sin^{-1} x) = x; -1 \le x \le 1$

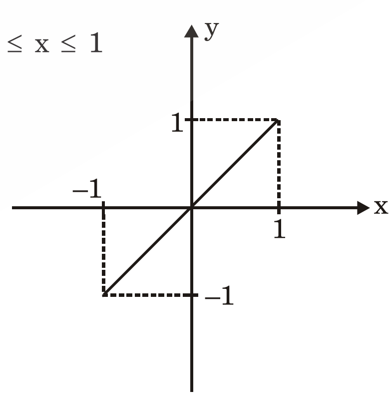


Illustration 48

Draw the following graphs :

(a) $y = \tan^{-1}(\tan x)$

(b) $y = \sec(\sec^{-1} x)$

Solution :

(a) given $y = \tan^{-1}(\tan x)$

Again here, first of all we will try to find out the domain of the function

We know $\tan^{-1} x$ will be valid for all $x \in R$

& $\tan x$ returns real values for $x \in R - (2n+1)\frac{\pi}{2}$

$\therefore \quad D_f \in R - (2n+1)\frac{\pi}{2}$

& Range will be according to the outer function which is $\tan^{-1}$ here

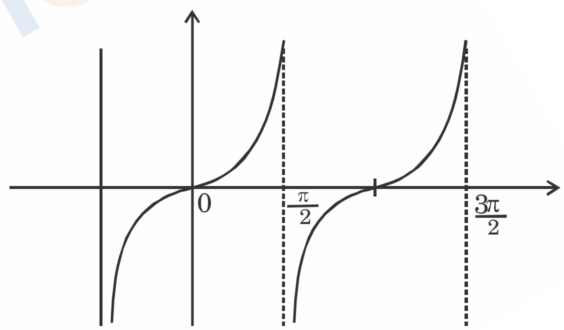
$\Rightarrow \quad R_f \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

We know the graph of $y = \tan x$

Which is discontinuous at $x \in (2n+1)\frac{\pi}{2}$

so $(2n+1)\frac{\pi}{2}$ for $n = 0, 1, 2 \ldots$

become its critical points.

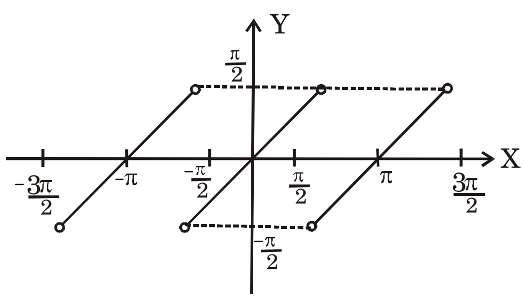


so for $\frac{-\pi}{2} < x < \frac{\pi}{2}$ $\tan^{-1}(\tan x) = x$

& for $\frac{\pi}{2} < x < \frac{3\pi}{2}$ $\tan^{-1}(\tan x) = x - \pi$

again according to principal value branch which $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ for $\tan^{-1} x$.

$$\therefore \quad y = \tan^{-1}(\tan x) = \begin{cases} x + \pi & \frac{-3\pi}{2} < x < \frac{-\pi}{2} \\ x & \frac{-\pi}{2} < x < \frac{\pi}{2} \\ x - \pi & \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$$



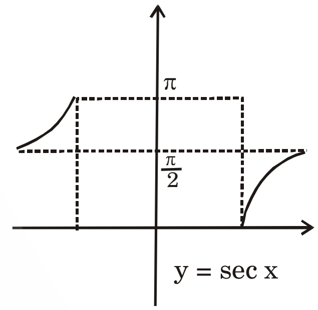
Note : $(2n+1)\frac{\pi}{2}$ points are not included as they are not part of domain.

(b) $y = \sec(\sec^{-1} x)$

we know the graph of $\sec^{-1} x$

$D_f \in (-\infty, -1] \cup [1, \infty)$

$R_f \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$



and for this range the sec function is also valid.

$\Rightarrow$ $D_f \in (-\infty, -1] \cup [1, \infty)$

& $R_f \in (-\infty, -1] \cup [1, \infty)$ (as equal to the range of normal sec x function)

$\therefore$ $y = \sec(\sec^{-1} x) = x$ (for the given domain)

Note : The funda of principal value branch comes only when the outer function is an inverse function because it is the property of inverse functions only.

So, drawing the graph now

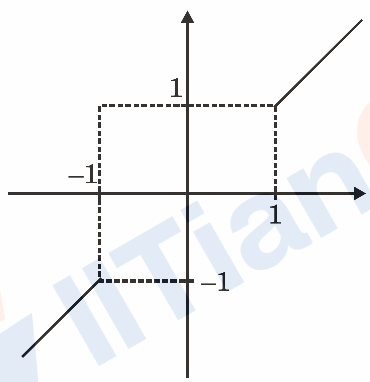


Illustration 49

Draw the following graphs :

(a) $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ **(b)** $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Solution : (a) given, $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

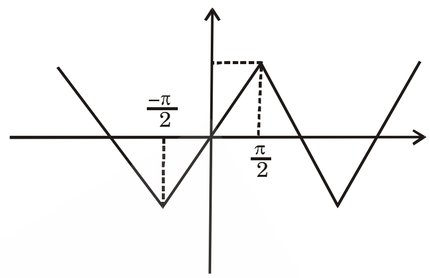
These kind of problems are solved by substitution by putting $x = \tan\theta$ & $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ the principal branch value for tan

we get $\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$ $\left[\text{since } \frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta\right]$

$= \sin^{-1}(\sin 2\theta)$

So, now the function becomes

$y = \sin^{-1}(\sin 2\theta)$



but we know the graph of $\sin^{-1}(\sin x)$ i.e. $y = \begin{cases} -x+\pi & \frac{-3\pi}{2} \le x < \frac{-\pi}{2} \\ x & \frac{-\pi}{2} \le x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \le x < \frac{3\pi}{2} \end{cases}$ replacing x by 2θ

replacing x by 2θ

$$y = \begin{cases} -2\theta + \pi & ; \frac{-3\pi}{2} \le 2\theta < \frac{-\pi}{2} \\ 2\theta & ; \frac{-\pi}{2} \le 2\theta < \frac{\pi}{2} \\ \pi - 2\theta & ; \frac{\pi}{2} \le 2\theta \le \frac{3\pi}{2} \end{cases}$$

$$\Rightarrow \quad y = \begin{cases} -2\theta + \pi & ; \frac{-3\pi}{4} \le \theta \le \frac{-\pi}{4} \\ 2\theta & ; -{}^{\pi}/_{4} \le \theta < {}^{\pi}/_{4} \\ \pi - 2\theta & ; \frac{\pi}{4} \le \theta < \frac{3\pi}{4} \end{cases}$$

and we substitute $x = \tan\theta$

$$\Rightarrow \theta = \tan^{-1} x \ \& \text{ also } \theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

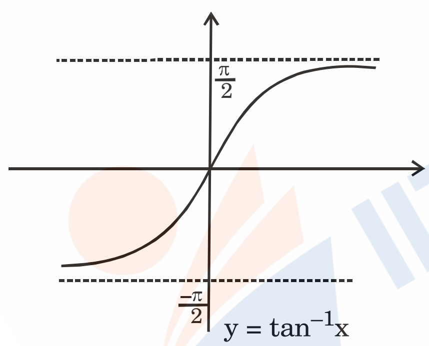
replacing the value of θ in above values

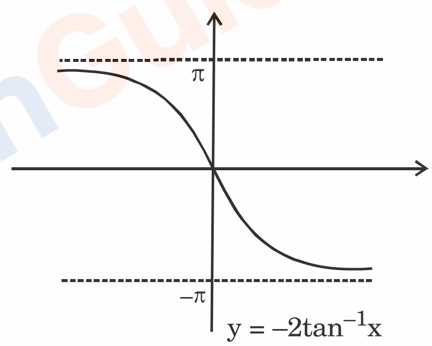
$$\Rightarrow \quad y = \begin{cases} -2\tan^{-1} x + \pi & ; -\frac{\pi}{2} < \tan^{-1} x \le \frac{-\pi}{4} \\ 2\tan^{-1} x & ; -\frac{\pi}{4} \le \tan^{-1} x < \frac{\pi}{4} \\ \pi - 2\tan^{-1} x & ; \frac{\pi}{4} \le \tan^{-1} x < \frac{\pi}{2} \end{cases}$$

$$\Rightarrow \quad y = \begin{cases} -2\tan^{-1}x + \pi & ; \quad \tan\left(-\frac{\pi}{2}\right) < x < \tan\left(-\frac{\pi}{4}\right) \\ 2\tan^{-1}x & ; \quad \tan\left(\frac{-\pi}{4}\right) \le x < \tan\left(\frac{\pi}{4}\right) \\ \pi - 2\tan^{-1}x & ; \quad \tan\frac{\pi}{4} \le x < \tan\left(\frac{\pi}{2}\right) \end{cases}$$

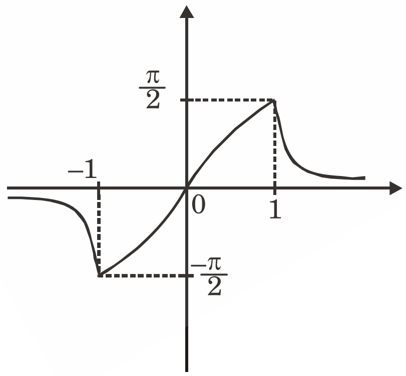
$$\Rightarrow \quad y = \begin{cases} -2\tan^{-1}x + \pi & ; \quad -\infty < x < -1 \\ 2\tan^{-1}x & ; \quad -1 \le x < 1 \\ \pi - 2\tan^{-1}x & ; \quad 1 \le x < \infty \end{cases}$$

Now we will draw the graph of the final function we know the graph of $\tan^{-1}x$, so drawing the final graph from it.





so the final graph is



(b) $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

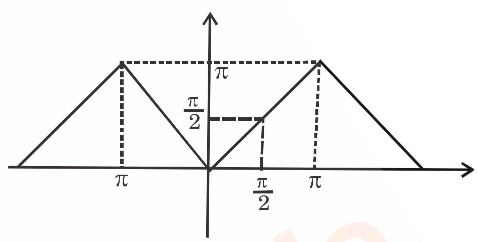
here putting $x = \tan\theta$

& $\theta \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$ {make a rule to write down constraint as they come

$\dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$

$\Rightarrow \quad y = \cos^{-1}(\cos 2\theta)$

$y = \cos^{-1}(\cos x) = \begin{cases} -x & , -\pi \le x < 0 \\ x & , 0 \le x \le \pi \end{cases}$



for those who do not know this graph, try to solve it on your own.

$\Rightarrow \quad y = \cos^{-1}(\cos(2\theta)) = \begin{cases} -2\theta & ; -\pi \le 2\theta \le 0 \\ 2\theta & ; 0 \le 2\theta < \pi \end{cases}$

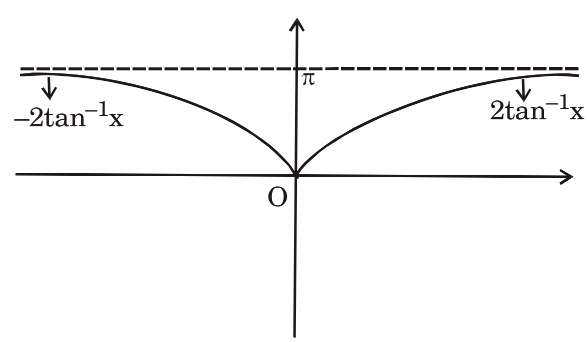
$\Rightarrow \quad y = \left\{\begin{matrix} -2\theta & ; \dfrac{-\pi}{2} \le \theta < 0 \\ 2\theta & ; 0 \le \theta < \dfrac{\pi}{2} \end{matrix}\right\}$

replacing θ by $\tan^{-1} x$

$\Rightarrow \quad \left\{\begin{matrix} -2\tan^{-1} x & ; & -\dfrac{\pi}{2} \le \tan^{-1} x < 0 \\ 2\tan^{-1} x & ; & 0 \le \tan^{-1} x < \dfrac{\pi}{2} \end{matrix}\right\}$

$\Rightarrow \quad y = \left\{\begin{matrix} -2\tan^{-1} x & ; -\infty \le x \le 0 \\ 2\tan^{-1} x & ; 0 < x < \infty \end{matrix}\right\}$

$\therefore$ Final graph is



Note :

•	If f (x) & g (x) are even	$\Rightarrow$ fog is an even function.
•	If f (x) & g (x) are odd	$\Rightarrow$ fog is an odd function.
•	If f (x) is even & g (x) is odd	$\Rightarrow$ fog is an even function.
•	If f (x) is odd & g (x) is even	$\Rightarrow$ fog is an even function.

MAPPING

Definition : Let X and Y be two non-empty sets. A subset f of X Y is called a function from X to Y iff to **each** $x \in X$, there exists a **unique** y in Y such that $(x, y) \in f$.

The other terms used for functions are "mappings", transformations" and "operators". We denote this mapping by

$f : X \to Y$ or $X \xrightarrow{f} Y$

It follows from the above definition that a relation X to Y is a function from X to Y iff

(i) to each $x \in X$ there exists a $y \in Y$ such that $(x, y) \in f$,

(ii) $(x, y_1) \in f$ and $(x, y_2) \in f \Rightarrow y_1 = y_2$.

The condition (i) ensures that to each x in X, f associates an element y in Y and condition (ii) guarantees that y is unique.

We call X, the domain of f and Y the co-domain of f. The unique element y in Y assigned to $x \in X$ is called the **image** of x under f or the **value** of f at x and is denoted by f (x). Also x is called a **pre-image** (or **inverse image**) of y. Note that there may be more than one pre-images of y. The **graph** of f is the subset of X Y defined by $\{(x, f(x)) : x \in X\}$ The **range** of f is the set of all images under f and is denoted by f [X]. Thus

$$f[X] = \{y \in Y : y = f(x) \text{ for some } x \in X\}$$
$$= \{f(x) : x \in X\}.$$

If $A \subset X$, then the set $\{f(x) : x \in A\}$ is called the image of A under f and is denoted by f [A]. If $B \subset Y$; then the set $\{x \in X : f(x) \in B\}$ is called the inverse image of B under f and is denoted by f^{-1} [B].

Many-one, one-one onto and into mappings.

Let $f : X \to Y$.

The mapping f is said to be **many-one** iff two or more different elements in X have the same f-image in Y. The mapping f is said to be **one-one** iff different elements in X have different f-images in Y i.e. if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ or equivalently, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. One-one mappings are also called **injection.** The mapping f is said to be **into** if there is **at least** one element in Y which is not the f-image of any element in X. Note that in this case the range of f is a proper subset of Y, that is f [X] $\subset$ Y and f [X] $\neq$ Y. The mapping f is said to be **onto** if every element in Y is the f-image of at least one element in X. In this case, the range of f is euqal to Y, that is f [X] = Y. Onto mapping are also called **surjection.** One-one and onto mappings are called **bijection.**

Illustration and introduction of words, 'one-one', 'many one' 'onto and into'.

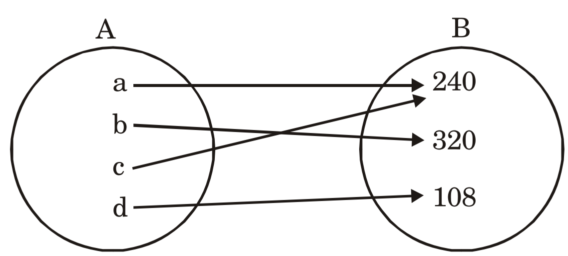
Let A be the set of books in a Library and B be the set of certain natural numbers. Let a, b, c, d, ... denote different books and let 240, 320, 108, 50 etc. denote some elements of the set B which correspond to the number of pages in the books.

Now choose f to be the correspondence which assigns to each book the number of pages contained in it ie. f : A $\rightarrow$ B.

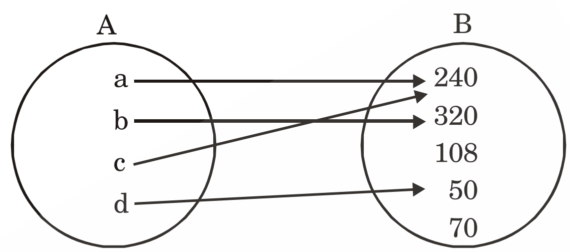
The following points should be clearly understood.

1. Each book $\in$ A is associated to some number $\in$ B (ie. the number of pages in that book). This number will be the image of the corresponding book.
2. Two or more books may be associated to the same number $\in$ B (i.e. Two or more of books may have the same number of pages).

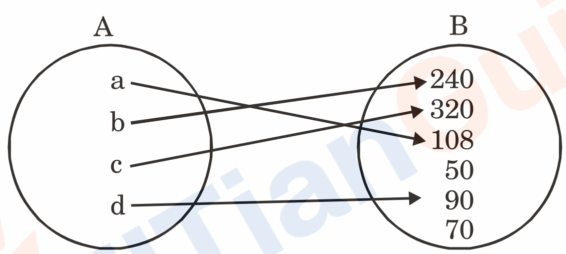
   In this case it will be termed as many one function or mapping as two or more elements $\in$ A will have the same image $\in$ B or an element $\in$ B will have more than one pre-image in A.
3. No book can be associated to different elements of B, i.e. the same book cannot have different number of pages in it i.e. each book is associated to a unique number $\in$ B i.e. the image of each and every book is **unique.**
4. If all the books $\in$ A are associated to different numbers $\in$ B i.e. all the books have different number of pages i.e. all the elements of A have different f images in B or an element of B has only one pre-image in A then this mapping is said to be **one-one** mapping.
5. There may be certain numbers in B which do not represent the number of pages of any of the books $\in$ A then the mapping is said to be **into mapping** i.e. f is a mapping of A into B. In other words there is at least one element $\in$ B which is not the image of any element $\in$ A then f is a mapping from A into B. In this case the set of images i.e. range of f is a subset of B.
6. Now suppose each number $\in$ B represents the number of pages of at least one book $\in$ to A then the mapping f is said to be **onto mapping** i.e. f **: A 'onto' B.** In this case each and every element of set B is the image of at least one element in A. The set B i.e. co-domain is completely covered by the f images of the domain A and consequently f (A) i.e. the set of images = B.
7. **Many-one onto mapping.** When two or more books $\in$ A have the same number of pages i.e. the same image $\in$ B (i.e. many-one) and also each and every number $\in$ B represent the number of pages of at least one book (i.e. onto) then f : A $\rightarrow$ B is a many-one onto mapping.



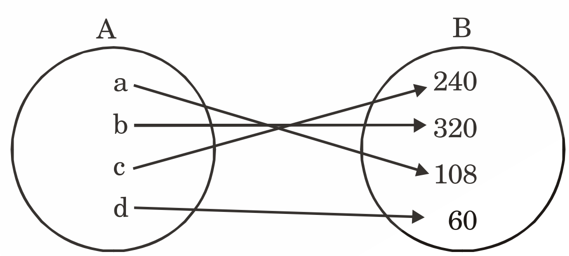
8. **Many-one into mapping.** When two or more books $\in$ A have the same number of pages i.e. the same image $\in$ B (i.e. many-one) and there are certain numbers $\in$ B which do not represent the number of pages of any of the books $\in$ A i.e. (into) then f : A $\rightarrow$ B is a many-one into mapping.



9. **One-one into mapping.** When all the books $\in$ A are having different number of pages i.e. they are associated to different numbers $\in$ B (one-one) and there are certain numbers $\in$ B which do not represent the number of pages of any of the books $\in$ A i.e. (into) then f : A $\rightarrow$ B is a one-one into mapping.



10. **One-one onto mapping.** When all the books $\in$ A are having different number of pages i.e. they are associated to diferent numbers $\in$ B (one-one) and there is no number $\in$ B which does not represent the number of pages of a book $\in$ A i.e. each and every number $\in$ B represents the number of pages of a certain book $\in$ A (onto) then f : A $\rightarrow$ B is a one-one onto mapping. This is also called **Bijection.**



How to Decide That a Relation is a mapping?

(1) Draw graph of y = f (x)

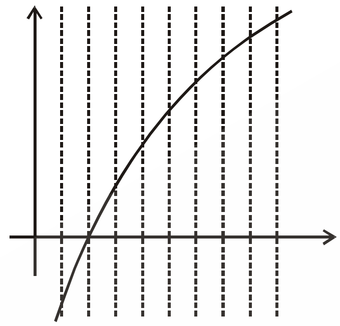
(2) Draw vertical lines in domain

(3) Make pt(s) of intersection b/w vertical lines and graph.

(4) If every vertical line has exactly one pt. of intersection then f (x) is a mapping.

e.g. $y = \log x\ (R \rightarrow R)$

not a mapping.



Type of Mappings :

(1) Injective mapping = one-one mapping – Non- (many-one mapping).

(2) Surjective mapping = onto mapping – Non (into mapping).

(3) Bijective mapping = inverse mapping (invertible)

Injective Mapping :

Every element in co-domain should have at most one pre-image.

Checking injective mapping :

(1) By inspection

eg. $y = x^2 = 1$, $y = 0$ at $x = \pm 1$

non-injective (many one mapping)

(2) Graphical Approach :

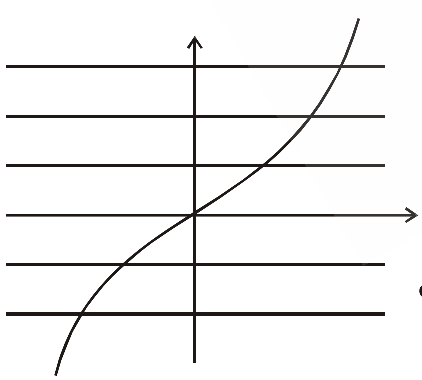
Steps : (i) Draw graph of f (x) in domain and co-domain.

(ii) Draw horizontal lines in co-domain.

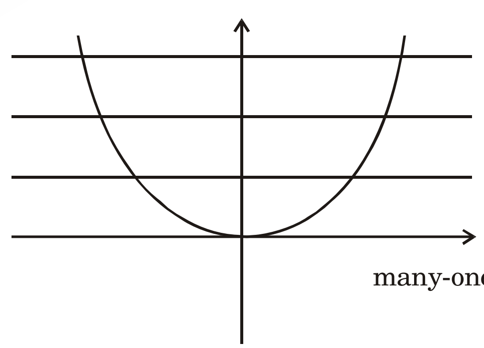
(iii) Mark pts. of intersection b/w graph.

(iv) If every horizontal line has atmost one pt. of intersection (0 or 1) with the graph then mapping is injetive or else many one mapping.

eg. $y = \tan x$ $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow R$ $y = x^2$



one-one



many-one

(3) Derivative method. (for continuous functions)

If f (x) has no pt. of (l max. or l-min) in domain i.e. either increasing/decreasing in domain then mapping is injective (one-one).

eg. $y = \log(\log x)$ $[1, \infty) \to R$

$$\frac{dy}{dx} = \frac{1}{\log_x}\frac{1}{x} \text{ is } > 0$$ for $x \in (1, \infty)$ one-one mapping (increasing function)

e.g $y = \log(\log(\log_x)$ $(e, \infty) \to R$

$$\frac{dy}{dx} = \frac{1}{\log(\log_x)}\frac{1}{\log_x} \times \frac{1}{x} > 0$$ $x \in (e, \infty)$

Surjective Mapping :

Every element is co-domain is a paired element or every element in co-domain has at least one pre image.

or (codomain = Range)

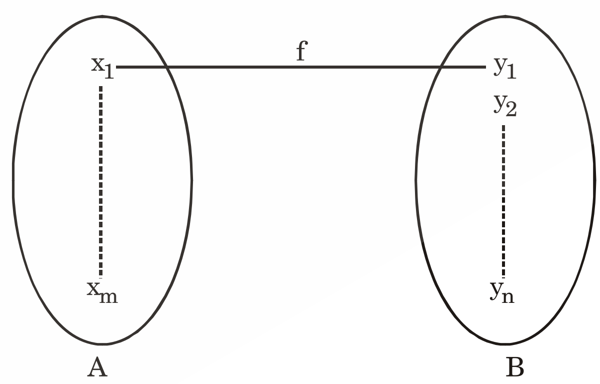
Bijection Mapping :

For bijection, function has to be both injective and surjective

Method to find no. of mappings

(1) Number of one-one mappings

If A & B are 2 finite sets having m & n elements respectively then, the number of one-one functions from A to B are–



There are m elements in set A & n in set B. For one-one mapping to occur $n \geq m$

Now, x_1 can take n images

x_2 can take $(n - 1)$ images, removing the one used by x_1 already.

Similarly x_3

x_n can take $(n - m + 1)$ images

$\therefore$ Total number of mappings $= n \quad (n - 1)(n - 2) \ldots\ldots (n - m + 1)$

$$= {}^nP_m$$

$$\text{Mapping possible} \begin{cases} {}^nP_m & ; \text{ if } n \geq m \\ 0 & ; \text{ if } n < m \end{cases}$$

2. Number of onto functions

For this case you can just remember the formula or see a simple example given in your package. For surjection from A to B, where A contains m and B contains n elements

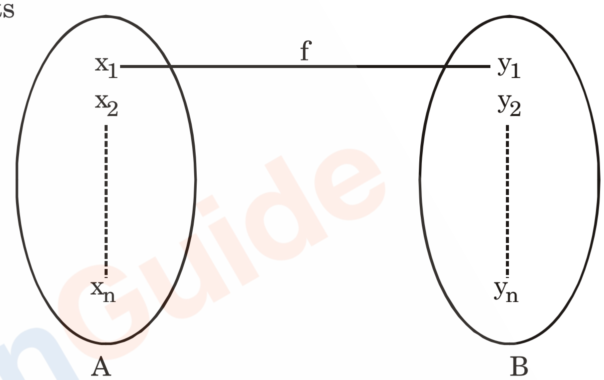
The formula is $\sum_{r=1}^{n}(-1)^{n-r}\ {}^{n}C_{r}\,(r)^{m}$

3. Number of bijection mappings

For a mapping to be bijective, it must be both,

one-one & onto.

i.e. both sets should have same number of elements



$\therefore$ x_1 can take (n) images

x_2 can ...... (n – 1) images

x_n can take 1 image

$\therefore$ Total number of mappings

= n (n – 1) (n – 2) ... 1

= n!

Illustration 50

Let A = {x : – 1 ≤ x ≤ 1} = B be a mapping f : A → B. For each of the following functions from A to B, find whether it is surjective or bijective.

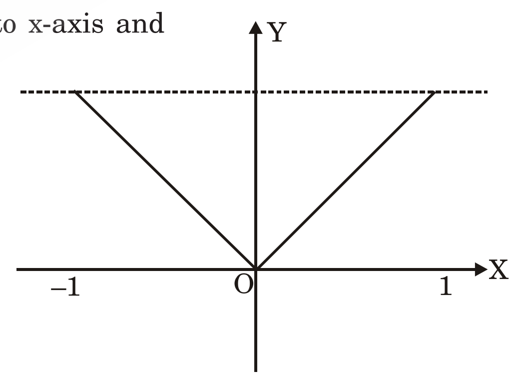
(a) f (x) = |x| **(b)** f (x) = |x| **(c)** f (x) = x^3

(d) f (x) = [x] **(e)** $f(x) = \sin\frac{\pi x}{2}$

Solution :

(a) f (x) = |x|

Graphically;



Which shows many one, as the straight line is parallel to x-axis and cuts at two points. Here range for f (x) ∈ [0, 1]

Which is clearly subset of co-domain.

i.e. [0, 1] ⊆ [– 1, 1]

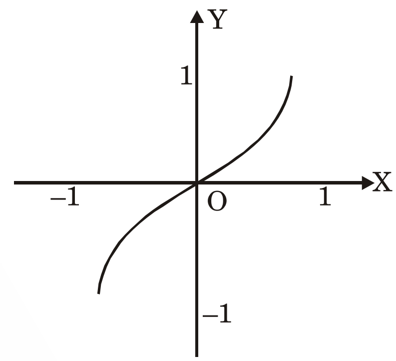
Thus, into

Hence, function is many-one-into.

$\therefore$ **neither injective nor surjective.**

(b) $f(x) = x\,|x|$,

Graphically,



The graph shows f (x) is one-one,

as the straight line parallel to x-axis cuts only at one point.

Here, range

$f(x) \in [-1, 1]$

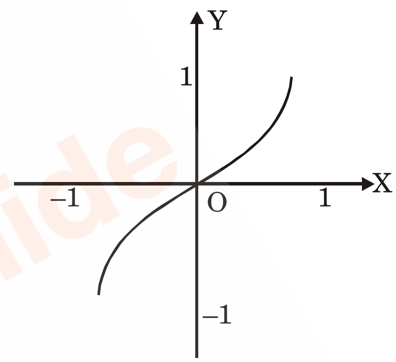
Thus, range = co-domain

Hence, onto.

Therefore, f(x) is one-one onto or **(bijective).**

(c) $f(x) = x^3$,

Graphically;



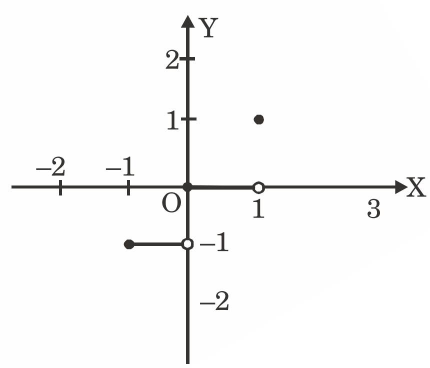
Graph shows f(x) is one-one onto

(i.e. **bijective**).

[as explained in above example].

(d) $f(x) = [x]$,

Graphically;



which shows f(x) is many-one, as the straight line parallel to x-axis meets at more than one point.

Here, range

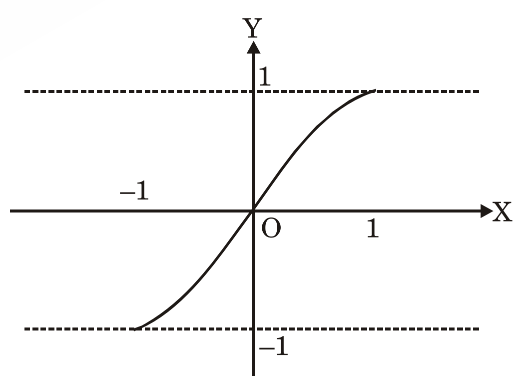
$f(x) \in \{-1, 0, 1\}$

which shows into as range co-domain.

Hence, **many-one-into.**

(e) $f(x) = \sin\dfrac{\pi x}{2}$

Graphically,



which shows f(x) is one-one and onto as range.

= co-domain.

Therefore, f(x) is **bijective.**

Illustration 51

Find number of surjections from A to B where

A = {1, 2, 3, 4}, B = {a, b} **[IIT 2000]**

Solution :

Number of surjection from A to B

$$= \sum_{r=1}^{2} (-1)^{2-r}\ {}^{2}C_{r}\,(r)^{4}$$

$$= (-1)^{2-1}\ {}^{2}C_{1}\ (1)^{4} + (-1)^{2-2}\ {}^{2}C_{2}(2)^{4}$$

$$= -2 + 16$$

$$= 14$$

Therefore, number of onto mapping from A to B = **14.**

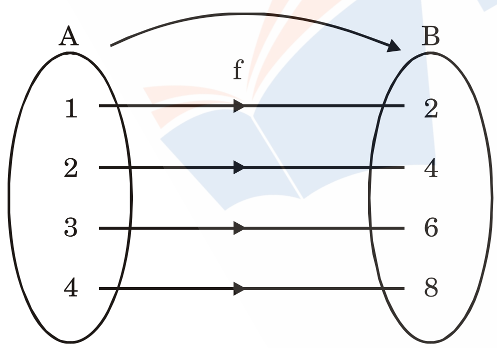
Alter : Total number of mapping from A to B is 2^4 of which two function f(x) = a for all $x \in A$ and g(x) = b for all $x \in A$ are not surjective.

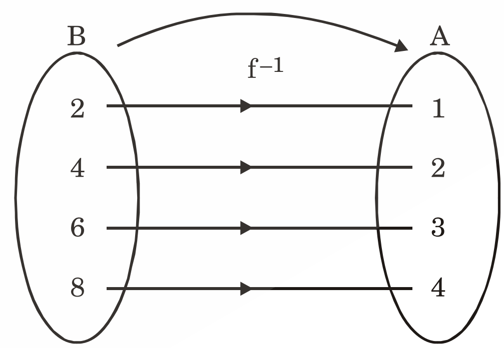
Thus, total number of surjection from A to B $= 2^4 - 2$

$= 14$

INVERSE OF FUNCTION

Let $f : A \to B$ be a one-one and onto function then there exists a unique function.





$g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x,\ \forall x \in A$ and $y \in B$.

Then g is said to be inverse of f.

Thus, $g = f^{-1} : B \to A = \{(f(x), x)]\ (x, f(x)) \in f\}$

Let us consider one-one function with domain A and range B.

Where A = {1, 2, 3, 4} and B = {2, 4, 6, 8} and $f : A \to B$ is given by f(x) = 2x, then write f and f^{-1} as a set of ordered pairs.

Here, member $y \in B$ arises from one and only one member $x \in A$.

So, $f = \{(1, 2,) (2, 4) (3, 6) (4, 8)\}$

and $f^{-1} = \{(2, 1) (4, 2) (6, 3) (8, 4)\}$

Note in above function

Domain of $f = \{1, 2, 3, 4\}$ = range of f^{-1}

Range of $f = (2, 4, 6, 8\}$ = domain of f^{-1}.

Which represents for a function to have its inverse it must be **one-one onto** or **(bijective).**

Method to find Inverse

- First of all we have to check whether the function is bijective, i.e. one-one & onto both, or not.
- If the function is bijective, then for $y = f(x)$ get
  1. $x = f(y)$
  2. put x as $f^{-1}(y)$ $\{$as $y = f(x) \Rightarrow f^{-1}(y) = x\}$
  3. replace y by x on right hand side

This is your inverse function

We can also use the following formula for finding the inverse.

$f\,[f^{-1}(x)] = x$

Illustration 52

If $f : [1, \infty) \rightarrow [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$ then find $f^{-1}(x)$. (assume bijective).

Solution :

Let $y = f(x)$

$$\therefore \quad y = \frac{x^2+1}{x} \quad \Rightarrow \quad x^2 - xy + 1 = 0$$

$$\Rightarrow \quad x = \frac{y \pm \sqrt{y^2-4}}{2}$$

$$\Rightarrow \quad f^{-1}(y) = \frac{y \pm \sqrt{y^2-4}}{2} \quad \{\text{as } f(x) = y \Rightarrow x = f^{-1}(y)\}$$

$$\Rightarrow \quad f^{-1}(x) = \frac{x \pm \sqrt{y^2-4}}{2}$$

Since, range of inverse function is $[1, \infty)$, therefore, neglecting negative sign, we have,

$$f^{-1}(x) = \frac{x + \sqrt{x^2-4}}{2}$$

Illustration 53

Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \dfrac{e^x - e^{-x}}{2}$. Is $f(x)$ invertible? If so, find its inverse.

Solution : Let us check for invertibility of $f(x)$:

(a) One-one : Here, $f'(x) = \dfrac{e^x + e^{-x}}{2}$

$\Rightarrow$ $f'(x) = \dfrac{e^{2x}+1}{2e^x}$ which is strictly increasing as $e^{2x} > 0$ for all x.

Thus, one-one.

(b) Onto : Let $y = f(x)$

$\Rightarrow$ $y = \dfrac{e^x - e^{-x}}{2}$ where y is strictly monotonic.

Hence, range of $f(x) = (f(-\infty), f(\infty))$

$\Rightarrow$ range of $f(x) = (-\infty, \infty)$

So, range of $f(x)$ = co-domain.

Hence, $f(x)$ is one-one and onto.

(c) To find f^{-1} : $y = \dfrac{e^{2x}-1}{2e^x}$

$\Rightarrow$ $e^{2x} - 2e^x y - 1 = 0$ $\Rightarrow$ $e^x = \dfrac{2y \pm \sqrt{4y^2+4}}{2}$

$\Rightarrow$ $x = \log\left(y \pm \sqrt{y^2+1}\right)$

$\Rightarrow$ $f^{-1}(y) = \log\left(y \pm \sqrt{y^2+1}\right)$ [as $f(x) = y \Rightarrow x = f^{-1}(y)$]

Since, $e^{f^{-1}(x)}$ is always positive.

So, neglecting negative sign.

Hene, $f^{-1}(x) = \log\left(x + \sqrt{x^2+1}\right)$

Illustration 54

Let $f : [1/2, \infty) \to [3/4, \infty)$, where $f(x) = x^2 - x + 1$. Find the inverse of $f(x)$.

Hence, solve the equation $x^2 - x + 1 = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$

Solution :

(a) $f(x) = x^2 - x + 1$

$\Rightarrow \quad f(x) = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$ which is clearly one-one and onto in given domain and co-domain.

(b) Thus, its inverse can be obtained.

let $\quad f(x) = y$

$$\Rightarrow \quad y = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\Rightarrow \quad x - \frac{1}{2} = \pm\sqrt{y - \frac{3}{4}}$$

$$\Rightarrow \quad x = \frac{1}{2} \pm \sqrt{y - \frac{3}{4}} \qquad [f(x) = y \Rightarrow x = f^{-1}(y)]$$

$$\Rightarrow \quad f^{-1}(y) = \frac{1}{2} + \sqrt{y - \frac{3}{4}} \qquad \text{[neglecting – ve sign as always + ve.]}$$

$$\Rightarrow \quad f^{-1}(x) = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$$

(c) **To solve :** $x^2 - x + 1 = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$, as $f(x) = f^{-1}(x)$ has only one solution.

i.e. $\quad f(x) = x$

$\Rightarrow \quad x^2 - x + 1 = x$

$\Rightarrow \quad x^2 - 2x + 1 = 0 \quad \Rightarrow \quad (x - 1)^2 = 0$

$x = 1$ is the required solution.

Properties of inverse of a function

1. The inverse of bijection is unique.
2. The inverse of bijection is also bijection.
3. If $f : A \to B$ is bijection and $g : B \to A$ is inverse of f.

   Then $fog = I_B$ and $gof = I_A$.

   where, I_A and I_B are identity functions on the sets A and B respectively.
4. If $f : A \to B$ and $g : B \to C$ are two bijections, then $gof : A \to C$ is bijection and $(gof)^{-1} = (f^{-1} og^{-1})$.
5. $fog \neq gof$ but if, $fog = gof$ then either $f^{-1} = g$ or $g^{-1} = f$ also.

   $(fog)(x) = (gof)(x) = (x)$.

Illustration 55

Let g (x) be the inverse of f(x) and $f'(x) = \frac{1}{1+x^3}$. Then find g´(x) in terms of g(x).

Solution : We know, if g(x) is inverse of f(x)

$$\Rightarrow \quad g\{f(x) = (x)$$

$$\Rightarrow \quad g'(f(x)), f'(x) = 1$$

$$\Rightarrow \quad g'\{f(x)\} = \frac{1}{f'(x)} = 1 + x^3$$

$$\Rightarrow \quad g'\{f(g(x))\} = 1 + (g(x))^3$$

$$\Rightarrow \quad g'(x) = 1 + (g(x))^3 \quad [\therefore f(g(x)) = x]$$

SOME SPECIAL TYPE OF QUESTIONS

(I) Finding the number of solutions using graph

Illustration 56

Find the number of solutions of :

$$7^{|x|}(|5 - |x||) = 1$$

Solution :

In such type of questions we try to find the number of intersections of 2 curves.

We can write the equation as

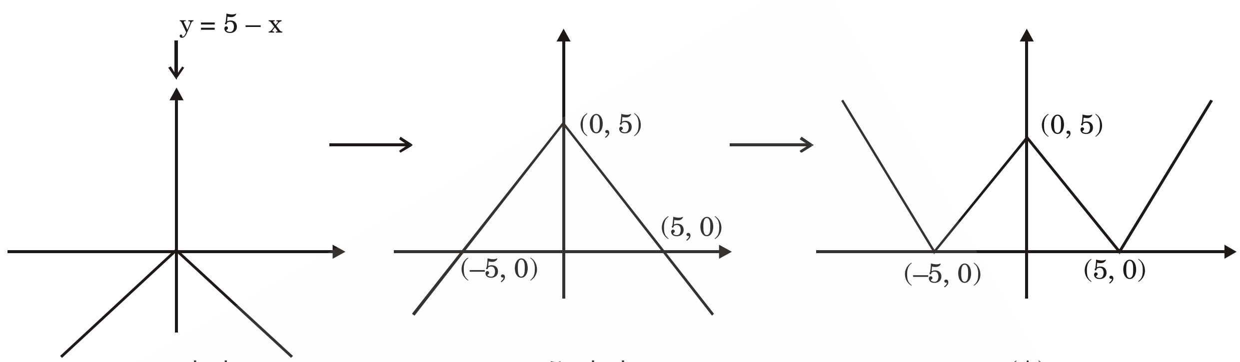
$$(|5 - |x||) = 7^{-|x|}$$

We did this to get 2 curves. Now we will draw LHS & RHS separately.

for y = $|5 - |x||$

using transformations 1. $|x| \to (x)$

2. $|f(x)| \to f(x)$

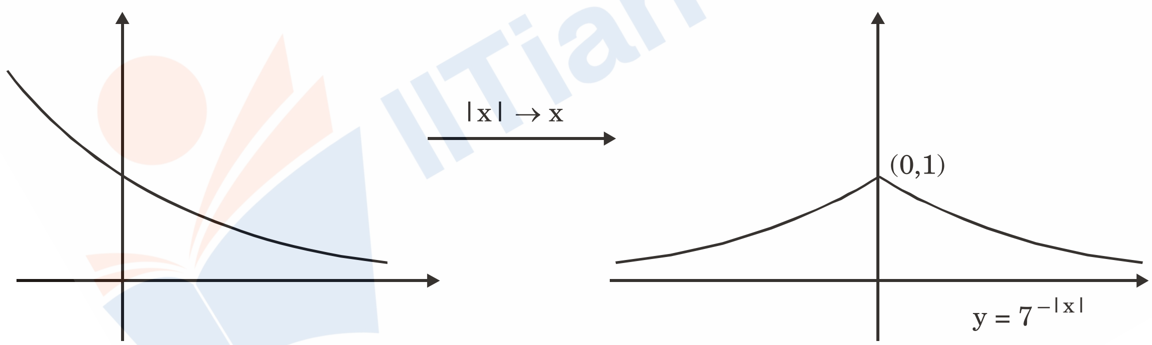


$y = -|x|$ $y = 5 - |x|$ (A)

now drawing $y = 7^{-|x|}$

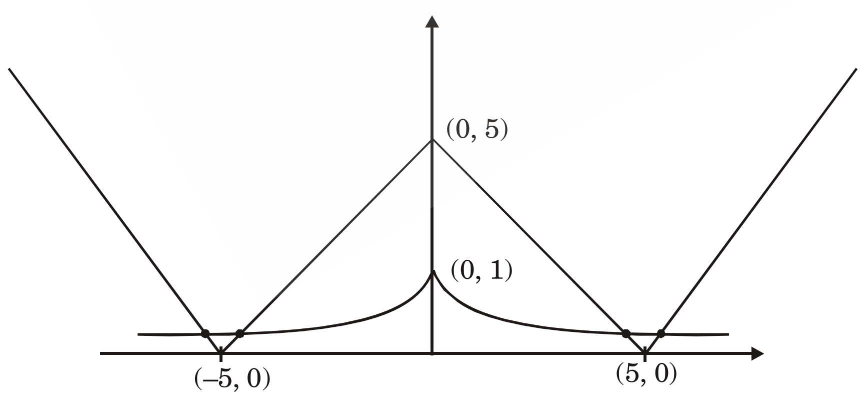
using $y = \left(\frac{1}{7}\right)^{|x|}$ & using $|x| \to x$

& we know that for a^x, $0 < a < 1$ the graph is



(B)

Combining both graphs (A) & (B)



Points of intersection = 4

$\therefore$ No. of solutions of the equation = 4

You can see how easy a question becomes if u are comfortable with graphs, solving this question algebraically could have been a bit confusing.

(II) To find the curve of $f(x)$ = max. $\{g(x), h(x)....\}$

In this type we draw the curves of all functions like $g(x)$, $h(x)$ ... and then we choose the part of the curve which is at top (has max value of y) with respect to all other curves in that region.

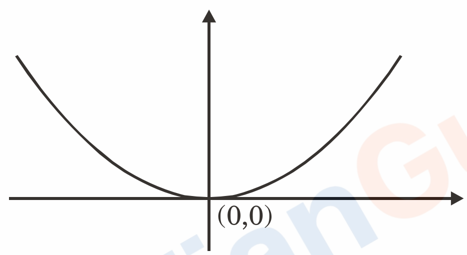
Illustration 57

Find the function/curve of $f(x) = \max.\{x^2, (1-x)^2, 2x(1-x)\}$

Solution :

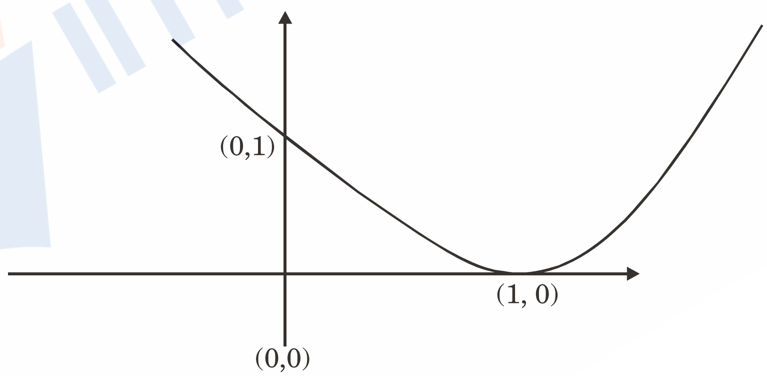
First we will draw the graphs of each of the function x^2, $(1-x)^2$, $2x(1-x)$

for $y = x^2$



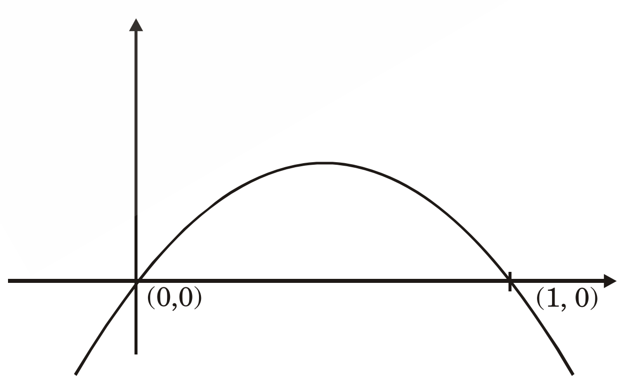
$y = (1-x)^2$

$= (x-1)^2$

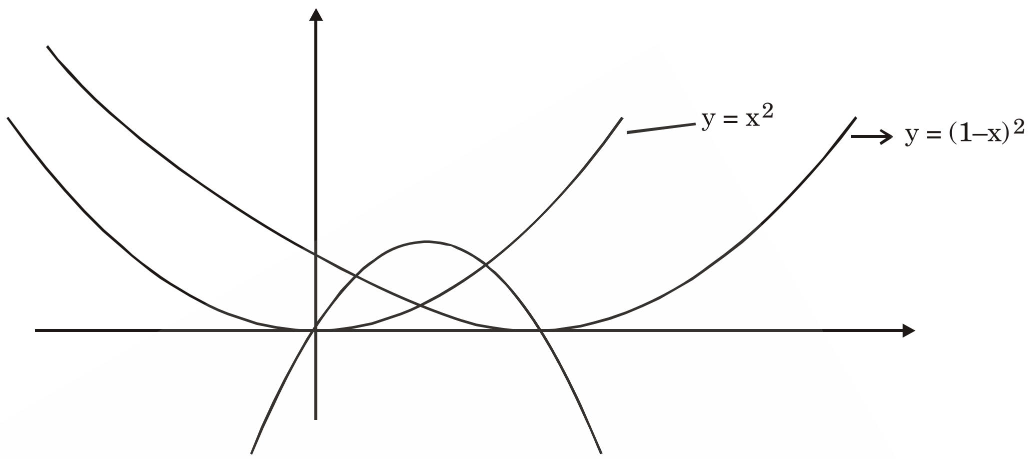


$y = 2x(1-x)$

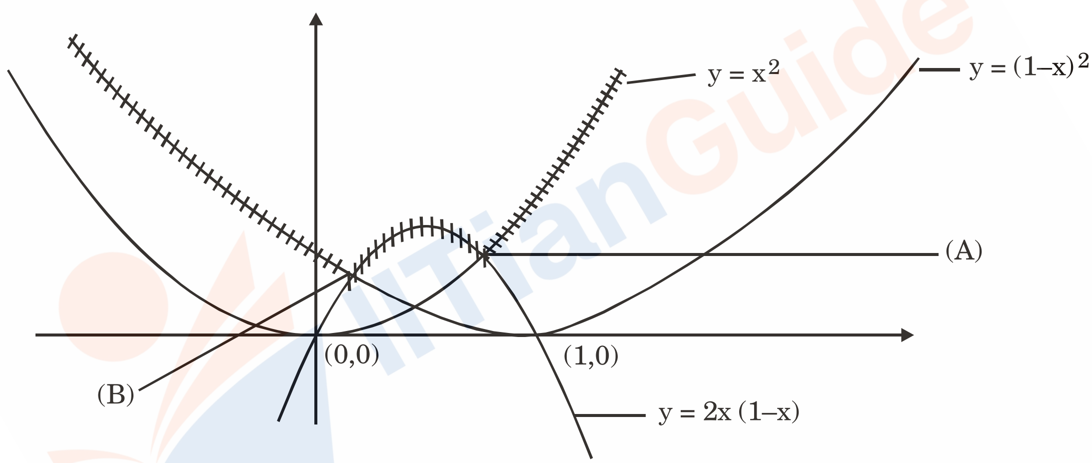
$= -2x^2 + 2x$



Now we will combine all the curves in one curve



Now mark the points which are above other curves



We just now have to find the intersection points (A) & (B) for (A) its between

curves $\quad y = 2x\,(1 - x)$ & $y = x^2$

equating $\quad 2x\,(1 - x) = x^2$

as we know the other point, i.e. $x = 0$

$$2 - 2x = x$$

$\Rightarrow \quad x = 2/3 \qquad$ ... (i)

for B its between $\quad y = (1 - x)^2$ & $y = 2x\,(1 - x)$

cancelling $(1 - x)$ from both sides

$$1 - x = 2x$$

$\Rightarrow \quad x = \dfrac{1}{3} \qquad$ ... (ii)

$\therefore$ The required function becomes

$$f(x) = \begin{cases} (1-x)^2 & -\infty < x \le \frac{1}{3} \\ 2x(1-x) & \frac{1}{3} < x \le \frac{2}{3} \\ x^2 & \frac{2}{3} < x < \infty \end{cases}$$

(III) Function as Series :

In this type, the function is given in the form of a series i.e. it is dependent on the values previous to it. Or some such relation is given to you.

Let us solve some questions to understand the concept.

Illustration 58

Let f be a function satisfying $f(x + y) = f(x) + f(y)$ for all $x, y \in$ R. If $f(1) = k$ then find $f(n)$.

Solution :

We are given the value of $f(1) =$ i.e. $f(1) = k$

& $f(x + y) = f(x) + f(y)$ ... (i)

& we have to find $f(n)$

putting $x = 1$ & $y = 1$ in (i)

$$f(2) = f(1) + f(1) = 2f(1) = 2k$$

$$f(3) = f(2) + f(1) = 2k + k = 3k$$

$$f(4) = f(3) + f(1) = 3k + k = 4k$$

$$f(n) = f(n-1) + f(1) = (n-1)k + k = nk$$

$$f(n) = nk$$

TRICKY ONE

Illustration 59

If $af(x+1)+bf\left(\frac{1}{x+1}\right)=x,\ x\neq -1,\ a\neq b$ **then find the value of** $f(2)$**.**

Solution :

Given $af(x+1)+bf\left(\frac{1}{x+1}\right)=x$

rewriting it

$$af(x+1)+b\left(\frac{1}{x+1}\right)=(x+1)-1 \qquad \text{... (i)}$$

replacing (x + 1) by $\frac{1}{x+1}$

$$\Rightarrow \qquad af\left(\frac{1}{x+1}\right)+bf(x+1)=\left(\frac{1}{x+1}\right)-1 \qquad \text{... (ii)}$$

now doing a (i) – b (ii), we get

$$\left(a^2-b^2\right)f(x+1)=a(x+1)-\left(\frac{b}{x+1}\right)+(b-a)$$

putting $x = 1$ to get f (2)

$$(a^2-b^2)\ f(2) = 2a-\frac{b}{2}+(b-a)$$

$$= a+\frac{b}{2}$$

$$\Rightarrow \qquad f(2)=\frac{2a+b}{2\left(a^2-b^2\right)}$$